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# The Science Absolute of Space

*Independent of the Truth or Falsity of Euclid's
Axiom XI (which can never be
decided a priori).*

BY

J~~OHN~~ BOLYAI

Translated from the Latin

BY

Dr. GEORGE BRUCE HALSTED

PRESIDENT OF THE TEXAS ACADEMY OF SCIENCE

*FOURTH EDITION.*

VOLUME THREE OF THE NEOMONIC SERIES

# TRANSLATOR'S INTRODUCTION.

The immortal *Elements* of Euclid was already in dim antiquity a classic, regarded as absolutely perfect, valid without restriction.

Elementary geometry was for two thousand years as stationary, as fixed, as peculiarly Greek, as the Parthenon. On this foundation pure science rose in Archimedes, in Apollonius, in Pappus; struggled in Theon, in Hypatia; declined in Proclus; fell into the long decadence of the Dark Ages.

The book that monkish Europe could no longer understand was then taught in Arabic by Saracen and Moor in the Universities of Bagdad and Cordova.

To bring the light, after weary, stupid centuries, to western Christendom, an Englishman, Adelhard of Bath, journeys, to learn Arabic, through Asia Minor, through Egypt, back to Spain. Disguised as a Mohammedan student, he got into Cordova about 1120, obtained a Moorish copy of Euclid's *Elements,* and made a translation from the Arabic into Latin.

The first printed edition of Euclid, published in Venice in 1482, was a Latin version from the Arabic. The translation into Latin from the Greek, made by Zamberti from a MS. of Theon's revision, was first published at Venice in 1505.

Twenty-eight years later appeared the *editio princeps* in Greek, published at Basle in 1533 by John Hervagius, edited by Simon Grynaeus. This was for a century and three-quarters the only printed Greek text of all the books, and from it the first English translation (1570) was made by "Henricus Billingsley," afterward Sir Henry Billingsley, Lord Mayor of London in 1591.

And even to-day, 1895, in the vast system of examinations carried out by the British Government, by Oxford, and by Cambridge, no proof of a theorem in geometry will be accepted which infringes Euclid's sequence of propositions.

Nor is the work unworthy of this extraordinary immortality.

Says Clifford: "This book has been for nearly twenty-two centuries the encouragement and guide of that scientific thought which is one thing with the progress of man from a worse to a better state.

''The encouragement; for it contained a body of knowledge that was really known and could be relied on.

''The guide; for the aim of every student of every subject was to bring his knowledge of that subject into a form as perfect as that which geometry had attained.''

But Euclid stated his assumptions with the most painstaking candor, and would have smiled at the suggestion that he claimed for his conclusions any other truth than perfect deduction from assumed hypotheses.   In favor of the external reality or truth of those assumptions he said no word.

Among Euclid's assumptions is one differing from the others in prolixity, whose place fluctuates in the manuscripts.

Peyrard, on the authority of the Vatican MS., puts it among the postulates, and it is often called the parallel-postulate.   Heiberg, whose edition of the text is the latest and best (Leipzig, 1883-1888), gives it as the fifth postulate.

James Williamson, who published the closest translation of Euclid we have in English, indicating, by the use of italics, the words not in the original, gives this assumption as eleventh among the Common Notions.

Bolyai speaks of it as Euclid's Axiom XI. Todhunter has it as twelfth of the Axioms.

Clavius (1574) gives it as Axiom 13.

The Harpur Euclid separates it by forty-eight pages from the other axioms.

It is not used in the first twenty-eight propositions of Euclid. Moreover, when at length used, it appears as the inverse of a proposition already demonstrated, the seventeenth, and is only needed to prove the inverse of another proposition already demonstrated, the twenty-seventh.

Now the great Lambert expressly says that Proklus demanded a proof of this assumption because when inverted it is demonstrable.

All this suggested, at Europe's renaissance, not a doubt of the necessary external reality and exact applicability of the assumption, but the possibility of deducing it from the other assumptions and the twenty-eight propositions already proved by Euclid without it.

Euclid demonstrated things more axiomatic by far. He proves what every dog knows, that any two sides of a triangle are together greater than the third.

Yet after he has finished his demonstration, that straight lines making with a transversal equal alternate angles are parallel, in order to

prove the inverse, that parallels cut by a transversal make equal alternate angles, he brings in the unwieldy assumption thus translated by Williamson (Oxford, 1781):

"11. And if a straight line meeting two straight lines make those angles which are inward and upon the same side of it less than two right angles, the two straight lines being produced indefinitely will meet each other on the side where the angles are less than two right angles."

As Staeckel says, "it requires a certain courage to declare such a requirement, alongside the other exceedingly simple assumptions and postulates." But was courage likely to fail the man who, asked by King Ptolemy if there were no shorter road in things geometric than through his *Elements?* answered, "To geometry there is no special way for kings!"

In the brilliant new light given by Bolyai and Lobachevski we now see that Euclid understood the crucial character of the question of parallels.

There are now for us no better proofs of the depth and systematic coherence of Euclid's masterpiece than the very things which, their cause unappreciated, seemed the most noticeable blots on his work.

Sir Henry Savile, in his Praelectiones on Euclid, Oxford, 1621, p. 140, says: "In pul-cherrimo Geometriae corpore duo sunt naevi, duae labes . . ." etc., and these two blemishes are the theory of parallels and the doctrine of proportion; the very points in the Elements which now arouse our wondering admiration. But down to our very nineteenth century an ever renewing stream of mathematicians tried to wash away the first of these supposed stains from the most beauteous body of Geometry.

The year 1799 finds two extraordinary young men striving thus

"To gild refined gold, to paint the lily,
To cast a perfume o'er the violet."

At the end of that year Gauss from Braun-schweig writes to Bolyai Farkas in Klausen-burg (Kolozsvár) as follows: [Abhandlungen der Koeniglichen Gesellschaft der Wissen-schaften zu Goettingen, Bd. 22, 1877.]

"I very much regret, that I did not make use of our former proximity, to find out *more* about your investigations in regard to the first grounds of geometry; I should certainly thereby have spared myself much vain labor, and would have become more restful than any one, such

as I, can be, so long as on such a subject there yet remains so much to be wished for.

In my own work thereon I myself have advanced far (though my other wholly heterogeneous employments leave me little time therefor) but *the* way, which I have hit upon, leads not so much to the goal, which one wishes, as much more to making doubtful the truth of geometry.

Indeed I have come upon much, which with most no doubt would pass for a proof, but which in my eyes proves as good as *nothing*.

For example, if one could prove, that a rectilineal triangle is possible, whose content may be greater, than any given surface, then I am in condition, to prove with perfect rigor all geometry.

Most would indeed let that pass as an axiom; I not; it might well be possible, that, how far apart soever one took the three vertices of the triangle in space, yet the content was always under a given limit.

I have more such theorems, but in none do I find anything satisfying."

From this letter we clearly see that in 1799 Gauss was still trying to prove that Euclid's is the only non-contradictory system of geome-

try, and that it is the system regnant in the external space of our physical experience.

The first is false; the second can never be proven.

Before another quarter of a century, Bolyai János, then unborn, had created another possible universe; and, strangely enough, though nothing renders it impossible that the space of our physical experience may, this very year, be satisfactorily shown to belong to Bolyai János, yet the same is not true for Euclid.

To decide our space is Bolyai's, one need only show a single rectilineal triangle whose angle-sum measures less than a straight angle. And this could be shown to exist by imperfect measurements, such as human measurements must always be.  For example, if our instruments for angular measurement could be brought to measure an angle to within one millionth of a second, then if the lack were as great as two millionths of a second, we could make certain its existence.

But to prove Euclid's system, we must show that a triangle's angle-sum is *exactly* a straight angle, which nothing human can ever do.

However this is anticipating, for in 1799 it seems that the mind of the elder Bolyai, Bolyai Farkas, was in precisely the same state as

that of his friend Gauss.   Both were intensely
trying to prove what now we know is inde-
monstrable.    And perhaps Bolyai got nearer
than Gauss to the unattainable.  In his ''Kurzer
Grundriss eines Versuchs,'' etc., p. 46, we read:
''Koennten jede 3 Punkte, die nicht in einer
Geraden sind, in eine Sphaere fallen, so waere
das Eucl. Ax. XI. bewiesen.''   Frischauf calls
this ''das anschaulichste Axiom.''   But in his
Autobiography written in Magyar, of which
my Life of Bolyai contains the first transla-
tion ever made, Bolyai Farkas says:   ''Yet I
could not become satisfied with my different
treatments of the question of parallels, which
was ascribable to the long discontinuance of
my studies, or more probably it was due to
myself that I drove this problem to the point
which robbed my rest, deprived me of tran-
quillity.''

It is wellnigh certain that Euclid tried his
own calm, immortal genius, and the genius of
his race for perfection, against this self-same
question.   If so, the benign intellectual pride
of the founder of the mathematical school of
the greatest of universities, Alexandria, would
not let the question cloak itself in the obscuri-
ties of the infinitely great or the infinitely
small.   He would say to himself:   ''Can I prove

this plain, straightforward, simple theorem:
"those straights which are produced indefin-
itely from less than two right angles meet."
[This is the form which occurs in the Greek
of Eu. I. 29.]

Let us not underestimate the subtle power
of that old Greek mind.  We can produce no
Venus of Milo.  Euclid's own treatment of
proportion is found as flawless in the chapter
which Stolz devotes to it in 1885 as when
through Newton it first gave us our present
continuous number-system.

But what fortune had this genius in the fight
with its self-chosen simple theorem?  Was it
found to be deducible from all the definitions,
and the nine "Common Notions," and the five
other Postulates of the immortal Elements?
Not so.  But meantime Euclid went ahead
without it through twenty-eight propositions,
more than half his first book.  But at last
came the practical pinch, then as now the tri-
angle's angle-sum.

He gets it by his twenty-ninth theorem:  "A
straight falling upon two parallel straights
makes the alternate angles equal."

But for the proof of this he needs that re-
calcitrant proposition which has how long
been keeping him awake nights and waking

him up mornings?   Now at last, true man of
science, he acknowledges it indemonstrable by
spreading it in all its ugly length among his
postulates.

Since Schiaparelli has restored the astron-
omical system of Eudoxus, and Hultsch has
published the writings of Autolycus, we see
that Euclid knew surface-spherics, was famil-
iar with triangles whose angle-sum is more
than a straight angle.   Did he ever think to
carry out for himself the beautiful system of
geometry which comes from the contradiction
of his indemonstrable postulate; which exists
if there be straights produced indefinitely from
less than two right angles yet nowhere meet-
ing; which is real if the triangle's angle-sum
is less than a straight angle?

Of how naturally the three systems of geom-
etry flow from just exactly the attempt we
suppose Euclid to have made, the attempt to
demonstrate his postulate fifth, we have a most
romantic example in the work of the Italian
priest, Saccheri, who died the twenty-fifth of
October, 1733.   He studied Euclid in the edi-
tion of Clavius, where the fifth postulate is
given as Axiom 13.   Saccheri says it should
not be called an axiom, but ought to be dem-
onstrated.   He tries this seemingly simple

task; but his work swells to a quarto book of 101 pages.

Had he not been overawed by a conviction of the absolute necessity of Euclid's system, he might have anticipated Bolyai János, who ninety years later not only discovered the new world of mathematics but appreciated the transcendent import of his discovery.

Hitherto what was known of the Bolyais came wholly from the published works of the father Bolyai Farkas, and from a brief article by Architect Fr. Schmidt of Budapest ''Aus dem Leben zweier ungarischer Mathematiker, Johann und Wolfgang Bolyai von Bolya.'' Grunert's Archiv, Bd. 48, 1868, p. 217.

In two communications sent me in September and October 1895, Herr Schmidt has very kindly and graciously put at my disposal the results of his subsequent researches, which I will here reproduce. But meantime I have from entirely another source come most unexpectedly into possession of original documents so extensive, so precious that I have determined to issue them in a separate volume devoted wholly to the life of the Bolyais; but these are not used in the sketch here given.

Bolyai Farkas was born February 9th, 1775, at Bolya, in that part of Transylvania (Er-

dély) called Székelyföld.  He studied first at Enyed, afterward at Klausenburg (Kolozsvár), then went with Baron Simon Kemény to Jena and afterward to Goettingen.  Here he met Gauss, then in his 19th year, and the two formed a friendship which lasted for life.

The letters of Gauss to his friend were sent by Bolyai in 1855 to Professor Sartorius von Walterhausen, then working on his biography of Gauss.  Everyone who met Bolyai felt that he was a profound thinker and a beautiful character.

Benzenberg said in a letter written in 1801 that Bolyai was one of the most extraordinary men he had ever known.

He returned home in 1799, and in January, 1804, was made professor of mathematics in the Reformed College of Maros-Vásárhely. Here for 47 years of active teaching he had for scholars nearly all the professors and no-bility of the next generation in Erdély.

Sylvester has said that mathematics is poesy.

Bolyai's first published works were dramas.

His first published book on mathematics was an arithmetic:

Az arithmetica eleje.  8vo. i–xvi, 1–162 pp. The copy in the library of the Reformed Col-lege is enriched with notes by Bolyai János.

Next followed his chief work, to which he constantly refers in his later writings. It is in Latin, two volumes, 8vo, with title as·follows:

TENTAMEN | JUVENTUTEM STUDIOSAM | IN ELEMENTA MATHESEOS PURAE, ELEMEN-TARIS AC | SUBLIMIORIS, METHODO INTUI-TIVA, EVIDENTIA— | QUE HUIC PROPRIA, IN-TRODUCENDI. |   ·

CUM APPENDICE TRIPLICI. | Auctore Professore Matheseos et Physices Chemiaeque | Publ. Ordinario. | Tomus Primus. | *Maros Vasarhelyini.* 1832. | Typis Collegii Reformatorum per JOSEPHUM, et | SIMEONEM KALI de felsö Vist. | At the back of the title: Imprimatur. | M. Vásárhelyini Die | 12 Octobris, 1829. | Paulus Horváth m. p. | Abbas, Parochus et Censor | Librorum.

Tomus Secundus. | *Maros Vasarhelyini.* 1833. |

The first volume contains:

Preface of two pages: *Lectori salutem.*

A folio table: *Explicatio signorum.*

*Index· rerum* (I—XXXII). *Errata·* (XXXIII—XXXVII).

*Pro tyronibus prima vice legentibus notanda sequentia* (XXXVIII—LII).

*Errores* (LIII—LXVI).

*Scholion* (LXVII—LXXIV).

*Plurium errorum haud animadversorum numerus minuitur* (LXXV—LXXVI).

*Recensio per auctorem ipsum facta* (LXXVII—LXXVIII).

*Errores recentius detecti* (L X X V— XCVIII).

Now comes the body of the text (pages 1—502).

Then, with special paging, and a new title page, comes the immortal Appendix, here given in English.

Professors Staeckel and Engel make a mistake in their "Parallellinien" in supposing that this Appendix is referred to in the title of "Tentamen." On page 241 they quote this title, including the words "Cum appendice triplici," and say: "In dem dritten Anhange, der nur 28 Seiten umfasst, hat Johann Bolyai seine neue Geometrie entwickelt."

It is not a third Appendix, nor is it referred to at all in the words "Cum appendice triplici."

These words, as explained in a prospectus in the Magyar language, issued by Bolyai Farkas, asking for subscribers, referred to a real triple Appendix, which appears, as it

should, at the end of the book Tomus Secun-
dus, pp. 265–322.

The now world renowned Appendix by
Bolyai János was an afterthought of the
father, who prompted the son not ''to occupy
himself with the theory of parallels,'' as
Staeckel says, but to translate from the Ger-
man into Latin a condensation of his treatise,
of which the principles were discovered and
properly appreciated in 1823, and which was
given in writing to Johann Walter von Eck-
wehr in 1825.

The father, without waiting for Vol. II,
inserted this Latin translation, with separate
paging (1–26), as an Appendix to his Vol. I,
where, counting a page for the title and a
page ''Explicatio signorum,'' it has twenty-
six numbered pages, followed by two unnum-
bered pages of Errata.

The treatise itself, therefore, contains only
twenty-four pages — the most extraordinary
two dozen pages in the whole history of
thought!

Milton received but a paltry £5 for his
Paradise Lost; but it was at least plus £5.

Bolyai János, as we learn from Vol. II, p.
384, of *"Tentamen,"* contributed for the

printing of his eternal twenty-six pages, 104 florins 50 kreuzers.

That this Appendix was finished considerably before the Vol. I, which it follows, is seen from the references in the text, breathing a just admiration for the Appendix and the genius of its author.

Thus the father says, p. 452: Elegans est conceptus *similium,* quem J. B. *Appendicis Auctor* dedit.   Again, p. 489: *Appendicis Auctor,* rem acumine singulari aggressus, Geometriam pro omni casu absolute veram posuit; quamvis e magna mole, tantum summe necessaria, in Appendice hujus tomi exhibuerit, multis (ut tetraedri resolutione generali, pluribusque aliis disquisitionibus elegantibus) brevitatis studio omissis.

And the volume ends as follows, p. 502: Nec operae pretium est plura referre; quum res tota exaltiori contemplationis puncto, in ima penetranti oculo, tractetur in Appendice sequente, a quovis fideli veritatis purae alumno diagna legi.

The father gives a brief resumé of the results of his own determined, life-long, desperate efforts to do that at which Saccheri, J. H. Lambert, Gauss also had failed, to establish Euclid's theory of parallels *a priori.*

He says, p. 490: "'Tentamina idcirco quae olim feceram, breviter exponenda veniunt; ne saltem alius quis operam eandem perdat." He anticipates J. Delboeuf's "Prolégoménes philosophiques de la géométrie et solution des postulats," with the full consciousness in addition that it is *not* the solution,—that the final solution has crowned not his own intense efforts, but the genius of his son.

This son's Appendix which makes all preceding space only a special case, only a species under a genus, and so requiring a descriptive adjective, *Euclidean*, this wonderful production of pure genius, this strange Hungarian flower, was saved for the world after more than thirty-five years of oblivion, by the rare erudition of Professor Richard Baltzer of Dresden, afterward professor in the University of Giessen. He it was who first did justice publicly to the works of Lobachevski and Bolyai.

Incited by Baltzer, in 1866 J. Hoüel issued a French translation of Lobachevski's Theory of Parallels, and in a note to his Preface says: "M. Richard Baltzer, dans la seconde édition de ses excellents *Elements de Geometrie,* a, le premier, introduit ces notions exactes à la place qu'elles doivent occuper." Honor to

Baltzer! But alas! father and son were already in their graves!

Fr. Schmidt in the article cited (1868) says: "It was nearly forty years before these profound views were rescued from oblivion, and Dr. R. Baltzer, of Dresden, has acquired imperishable titles to the gratitude of all friends of science as the first to draw attention to the works of Bolyai, in the second edition of his excellent Elemente der Mathematik (1866–67). Following the steps of Baltzer, Professor Hoüel, of Bordeaux, in a brochure entitled, Essai critique sur les principes fondamentaux de la Géométrie élémentaire, has given extracts from Bolyai's book, which will help in securing for these new ideas the justice they merit."

The father refers to the son's Appendix again in a subsequent book, Urtan elemei kezdöknek [Elements of the science of space for beginners] (1850–51), pp. 48. In the College are preserved three sets of figures for this book, two by the author and one by his grandson, a son of János.

The last work of Bolyai Farkas, the only one composed in German, is entitled,

Kurzer Grundriss eines Versuchs

I. Die Arithmetik, durch zvekmässig kons-

truirte Begriffe, von eingebildeten und unend-
lich-kleinen Grössen gereinigt, anschaulich
und logisch-streng darzustellen.

II. In der Geometrie, die Begriffe der ger-
aden Linie, der Ebene, des Winkels allgemein,
der winkellosen Formen, und der Krummen,
der verschiedenen Arten der Gleichheit u. d.
gl. nicht nur scharf zu bestimmen; sondern
auch ihr Seyn im Raume zu beweisen: und da
die Frage, *ob zwey von der dritten geschnit-
tene Geraden, wenn die summe der inneren
Winkel nicht = 2R, sich schneiden oder
nicht?* neimand auf der Erde ohne ein Axiom
(wie Euklid das XI) aufzustellen, beantworten
wird; die davon unabhängige Geometrie ab-
zusondern; und eine auf die *Ja*—Antwort,
andere auf das *Nein* so zu bauen, das die
Formeln der letzten, auf ein Wink auch in der
ersten gültig seyen.

Nach ein lateinischen Werke von 1829, M.
Vásárhely, und eben daselbst gedruckten un-
grischen.

Maros Vásárhely 1851.   8vo. pp. 88.

In this book he says, referring to his son's
Appendix:  ''Some copies of the work pub-
lished here were sent at that time to Vienna,
to Berlin, to Goettingen. . . . From Goet-
tingen the giant of mathematics, who from

his pinnacle embraces in the same view the stars and the abysses, wrote that he was surprised to see accomplished what he had begun, only to leave it behind in his papers.''

This refers to 1832.   The only other record that Gauss ever mentioned the book is a letter from Gerling, written October 31st, 1851, to Wolfgang Boylai, on receipt of a copy of ''Kurzer Grundriss.''   Gerling, a scholar of Gauss, had been from 1817 Professor of Astronomy at Marburg.   He writes:   ''I do not mention my earlier occupation with the theory of parallels, for already in the year 1810–1812 with Gauss, as earlier 1809 with J. F. Pfaff I had learned to perceive how all previous attempts to prove the Euclidean axiom had miscarried.   I had then also obtained preliminary knowledge of your works, and so, when I first [1820] had to print something of my view thereon, I wrote it exactly as it yet stands to read on page 187 of the latest edition.

''We had about this time [1819] here a law professor, Schweikart, who was formerly in Charkov, and had attained to similar ideas, since without help of the Euclidean axiom he developed in its beginnings a geometry which he called Astralgeometry.   What he communicated to me thereon I sent to Gauss, who

then informed me how much farther already
had been attained on this way, and later also
expressed himself about the great acquisition,
which is offered to the few expert judges in
the Appendix to your book."

The "latest edition" mentioned appeared
in 1851, and the passage referred to is: "This
proof [of the parallel-axiom] has been sought
in manifold ways by acute mathematicians,
but yet until now not found with complete
sufficiency. So long as it fails, the theorem,
as all founded on it, remains a hypothesis,
whose validity for our life indeed is suffici-
ently proven by *experience,* whose *general,
necessary exactness,* however, could be
doubted without absurdity."

Alas! that this feeble utterance should have
seemed sufficient for more than thirty years
to the associate of Gauss and Schweikart, the
latter certainly one of the independent discov-
erers of the non-Euclidean geometry. But
then, since neither of these sufficiently real-
ized the transcendent importance of the mat-
ter to publish any of their thoughts on the
subject, a more adequate conception of the
issues at stake could scarcely be expected of
the scholar and colleague. How different with
Bolyai János and Lobachévski, who claimed

at once, unflinchingly, that their discovery marked an epoch in human thought so momentous as to be unsurpassed by anything recorded in the history of philosophy or of science, demonstrating as had never been proven before the supremacy of pure reason at the very moment of overthrowing what had forever seemed its surest possession, the axioms of geometry.

On the 9th of March, 1832, Bolyai Farkas was made corresponding member in the mathematics section of the Magyar Academy.

As professor he exercised a powerful influence in his country.

In his private life he was a type of true originality.   He wore roomy black Hungarian pants, a white flannel jacket, high boots, and a broad hat like an old-time planter's.   The smoke-stained wall of his antique domicile was adorned by pictures of his friend Gauss, of Schiller, and of Shakespeare, whom he loved to call the child of nature.   His violin was his constant solace.

He died November 20th, 1856.   It was his wish that his grave should bear no mark.

The mother of Bolyai János, née Arkosi Benkö Zsuzsanna, was beautiful, fascinating,

of extraordinary mental capacity, but always nervous.

János, a lively, spirited boy, was taught mathematics by his father. His progress was marvelous. He required no explanation of theorems propounded, and made his own demonstrations for them, always wishing his father to go on. "Like a demon, he always pushed me on to tell him more."

At 12, having passed the six classes of the Latin school, he entered the philosophic-curriculum, which he passed in two years with great distinction.

When about 13, his father, prevented from meeting his classes, sent his son in his stead. The students said they liked the lectures of the son better than those of the father. He already played exceedingly well on the violin.

In his fifteenth year he went to Vienna to K. K. Ingenieur-Akademie.

In August, 1823, he was appointed "sous-lieutenant" and sent to Temesvár, where he was to present himself on the 2nd of September.

From Temesvár, on November 3rd, 1823, János wrote to his father a letter in Magyar, of which a French translation was sent me by Professor Koncz József on February 14th,

1895.   This will be given in full in my life of
Bolyai; but here an extract will suffice:

*"My Dear and Good Father:*
   "I have so much to write about my new
inventions that it is impossible for the mo-
ment to enter into great details, so I write
you only on one-fourth of a sheet.   I await
your answer to my letter of two sheets; and
perhaps I would not have written you before
receiving it, if I had not wished to address to
you the letter I am writing to the Baroness,
which letter I pray you to send her.
   "First of all I reply to you in regard to the
binominal.
*   *   *   *   *   *   *   *   *
   "Now to something else, so far as space
permits.   I intend to write, as soon as I have
put it into order, and when possible to pub-
lish, a work on parallels.
   "At this moment it is not yet finished,
but the way which I have followed promises
me with certainty the attainment of the goal,
if it in general is attainable.   It is not yet
attained, but I have discovered such magnifi-
cent things that I am myself astonished at
them.
   "It would be damage eternal if they were

lost.   When you see them, my father, you yourself will acknowledge it.   Now I can not say more, only so much: *that from nothing I have created another wholly new world.*   All that I have hitherto sent you compares to this only as a house of cards to a castle.

"P. S.—I dare to judge absolutely and with conviction of these works of my spirit before you, my father; I do not fear from you any false interpretation (that certainly I would not merit), which signifies that, in certain regards, I consider you as a second self."

From the Bolyai MSS., now the property of the College at Maros-Vásárhely, Fr. Schmidt has extracted the following statement by János:

"First in the year 1823 have I pierced through the problem in its essence, though also afterwards completions yet were added.

"I communicated in the year 1825 to my former teacher, Herr Johann Walter von Eckwehr (later k. k. General) [in the Austrian Army], a written treatise, which is still in his hands.

"On the prompting of my father I translated my treatise into the Latin language, and

it appeared as *Appendix* to the *Tentamen,* 1832."

The profound mathematical ability of Bolyai János showed itself physically not only in his handling of the violin, where he was a master, but also of arms, where he was unapproachable.

It was this skill, combined with his haughty temper, which caused his being retired as Captain on June 16th, 1833, though it saved him from the fate of a kindred spirit, the lamented Galois, killed in a duel when only 19. Bolyai, when in garrison with cavalry officers, was provoked by thirteen of them and accepted all their challenges on condition that he be permitted after each duel to play a bit on his violin. He came out victor from his thirteen duels, leaving his thirteen adversaries on the square.

He projected a universal language for speech as we have it for music and for mathematics.

He left parts of a book entitled: Principia doctrinae novae quantitatum imaginariarum perfectae uniceque satisfacientis, aliaeque disquisitiones analyticae et analytico‑geometricae cardinales gravissimaeque; auctore

Johan. Bolyai de eadem, C. R. austriaco cas-
trensium captaneo pensionato.

Vindobonae vel Maros Vásárhelyini, 1853.

Bolyai Farkas was a student at Goettingen
from 1796 to 1799.

In 1799 he returned to Kolozsvár, where
Bolyai János was born December 18th, 1802.

He died January 27th, 1860, four years
after his father.

In 1894 a monumental stone was erected on
his long-neglected grave in Maros-Vásárhely
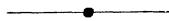by the Hungarian Mathematico-Physical So-
ciety.

# APPENDIX.

SCIENTIAM SPATII *absolute veram* exhibens:

*a veritate aut falsitate Axiomatis* XI *Euclidei*
*(a priori haud unquam decidenda) in-*
*dependentem:* adjecta ad casum fal-
sitatis, quadratura circuli
geometrica.

———•———

Auctore JOHANNE BOLYAI de eadem, Geometrarum
in Exercitu Caesareo Regio Austriaco
Castrensium Capitaneo.

══════

# EXPLANATION OF SIGNS.

*The straight AB* means the aggregate of all points situated in the same straight line with A and B.

*The sect AB* means that piece of the straight AB between the points A and B.

*The ray AB* means that half of the straight AB which commences at the point A and contains the point B.

*The plane ABC* means the aggregate of all points situated in the same plane as the three points (not in a straight) A, B, C.

*The hemi-plane ABC* means that half of the plane ABC which starts from the straight AB and contains the point C.

*ABC* means the smaller of the pieces into which the plane ABC is parted by the rays BA, BC, or the non-reflex angle of which the sides are the rays BA, BC.

ABCD (the point D being situated within $\angle$ ABC, and the straights BA, CD not intersecting) means the portion of $\angle$ ABC comprised between ray BA, sect BC, ray CD; while BACD designates the portion of the plane ABC comprised between the straights AB and CD.

$\perp$ is the sign of perpendicularity.

$\parallel$ is the sign of parallelism.

$\angle$ means angle.

rt. $\angle$ is right angle.

st. $\angle$ is straight angle.

$\cong$ is the sign of congruence, indicating that two magnitudes are superposable.

AB $\approxeq$ CD means $\angle$ CAB $= \angle$ ACD.

$x \doteq a$ means $x$ converges toward the limit $a$.

$\triangle$ is triangle.

$\odot r$ means the [circumference of the] circle of radius $r$.

area $\odot r$ means the area of the surface of the circle of radius $r$.

# THE SCIENCE ABSOLUTE OF SPACE.

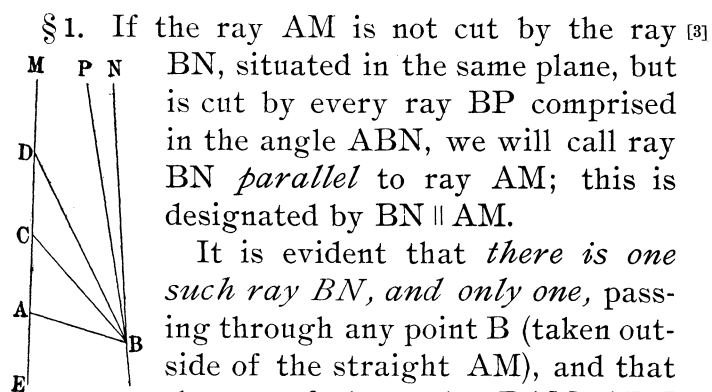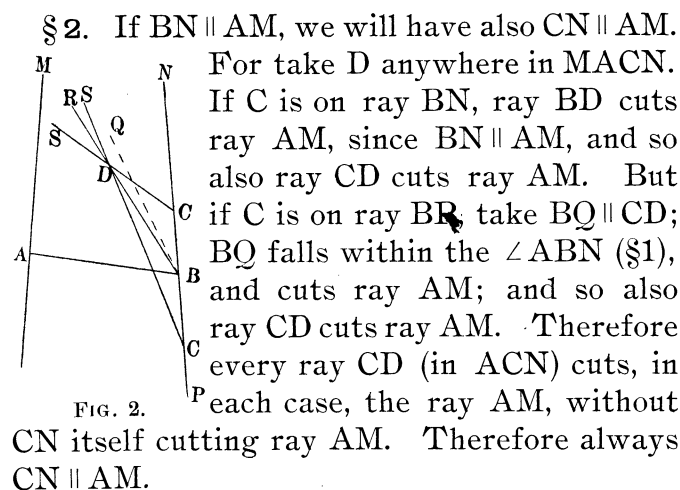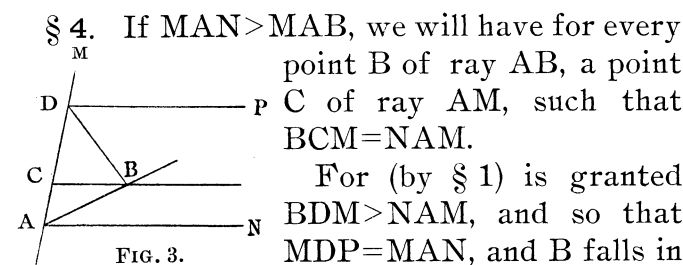§ 1. If the ray AM is not cut by the ray [3] BN, situated in the same plane, but is cut by every ray BP comprised in the angle ABN, we will call ray BN *parallel* to ray AM; this is designated by BN ‖ AM.

It is evident that *there is one such ray BN, and only one,* passing through any point B (taken outside of the straight AM), and that the sum of the angles BAM, ABN can not exceed a st. ∠; for in moving BC around B until BAM+ABC=st. ∠, somewhere ray BC *first* does not cut ray AM, and it is then BC ‖ AM. It is clear that BN ‖ EM, wherever the point E be taken on the straight AM (supposing in all such cases AM>AE).

If while the point C goes away to infinity on ray AM, always CD=CB, we will have constantly CDB=(CBD<NBC); but NBC≐0; and so also ADB≐0.

FIG. 1.

§2.  If BN ∥ AM, we will have also CN ∥ AM.



For take D anywhere in MACN. If C is on ray BN, ray BD cuts ray AM, since BN ∥ AM, and so also ray CD cuts ray AM.  But if C is on ray BR, take BQ ∥ CD; BQ falls within the ∠ ABN (§1), and cuts ray AM; and so also ray CD cuts ray AM.  Therefore every ray CD (in ACN) cuts, in each case, the ray AM, without CN itself cutting ray AM.  Therefore always CN ∥ AM.

Fig. 2.

§3.  (Fig. 2.)  If BR and CS and each ∥ AM, and C is not on the ray BR, then ray BR and ray CS do not intersect.  For if ray BR and ray CS had a common point D, then (§ 2) DR and DS would be each ∥ AM, and ray DS (§ 1) would fall on ray DR, and C on the ray BR (contrary to the hypothesis).

§4.  If MAN>MAB, we will have for every point B of ray AB, a point C of ray AM, such that BCM=NAM.



Fig. 3.

For (by § 1) is granted BDM>NAM, and so that MDP=MAN, and B falls in

NADP.  If therefore NAM is carried along AM until ray AN arrives on ray DP, ray AN will somewhere have necessarily passed through B, and some BCM=NAM.

§ 5.  If BN ‖ AM, there is on the straight [4] AM a point F such that FM ≏ BN. For by §1 is granted BCM>CBN; and if CE=CB, and so EC ≏ BC; evidently BEM<EBN.  The point P is moved on EC, the angle BPM always being called $u$, and the angle PBN always $v$; evidently $u$ is at first less than the corresponding $v$, but afterwards greater.  Indeed $u$ increases *continuously* from BEM to BCM; since (by §4) there exists *no* angle >BEM and <BCM, to which $u$ does not at some time become equal.  Likewise $v$ decreases continuously from EBN to CBN.  There is therefore on EC a point F such that BFM=FBN.
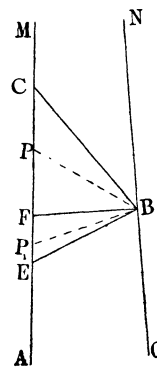
FIG. 4.

§6.  If BN ‖ AM and E anywhere in the straight AM, and G in the straight BN; then GN ‖ EM and EM ‖ GN.  For (by §1) BN ‖ EM, whence (by §2) GN ‖ EM.  If moreover FM ≏ BN (§5); then MFBN ≅ NBFM, and consequently (since BN ‖ FM) also FM ‖ BN, and (by what precedes) EM ‖ GN.

§ **7**.  If  BN  and  CP  are  each  ‖ AM,  and  C
not on the straight BN; also BN ‖ CP.
For  the  rays  BN  and  CP  do  not  in-
tersect (§3);  but  AM,  BN  and  CP
either  are  or  are  not  in  the  same
plane; and in the first case, AM either
is or is not within  BNCP.
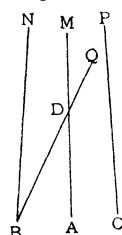
FIG. 5.     If AM, BN, CP are complanar, and
AM  falls  within  BNCP;  then  every  ray  BQ
(in  NBC)  cuts  the  ray  AM  in  some  point  D
(since  BN ‖ AM);  moreover,  since  DM ‖ CP
(§ 6),  the  ray  DQ  will  cut  the  ray  CP,  and  so
BN ‖ CP.
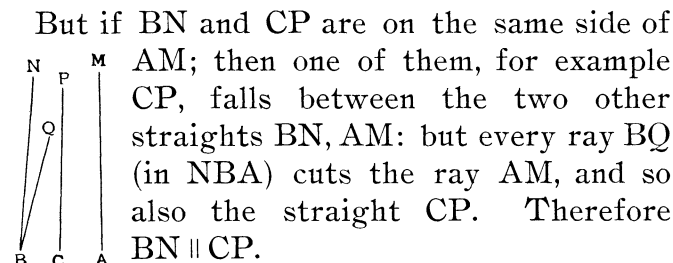
But if BN and CP are  on  the  same side of
AM;  then  one  of  them,  for  example
CP,  falls  between  the  two  other
straights BN, AM: but every ray BQ
(in  NBA)  cuts  the  ray  AM,  and  so
also  the  straight  CP.    Therefore
BN ‖ CP.

FIG. 6.     If  the  planes  MAB,  MAC  make
*an angle;* then  CBN  and  ABN  have  in  com-
mon  nothing  but  the  ray  BN,  while  the  ray
AM  (in  ABN)  and  the  ray  BN,  and  so  also
NBC  and  the  ray  AM  have  nothing  in  com-
mon.

But  hemi-plane  BCD,  drawn  through  any
ray  BD  (in  NBA),  cuts  the  ray  AM,  since  ray
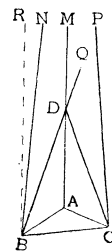
BQ cuts ray AM (as BN ∥ AM).
Therefore in revolving the hemi-plane [5]
BCD around BC until it *begins* to
leave the ray AM, the hemi-plane
BCD at last will fall upon the hemi-
plane BCN. For the same reason this
same will fall upon hemi-plane BCP.
Therefore BN falls in BCP. More-
over, if BR ∥ CP; then (because also AM ∥ CP)
by like reasoning, BR falls in BAM, and also
(since BR ∥ CP) in BCP. Therefore the
straight BR, being common to the two planes
MAB, PCB, of course is the straight BN, and
hence BN ∥ CP.*

FIG. 7.

If therefore CP ∥ AM, and B exterior to the
plane CAM; then the intersection BN of the
planes BAM, BCP is ∥ as well to AM as to CP.

§ 8. If BN ∥ and ≑ CP (or more briefly BN
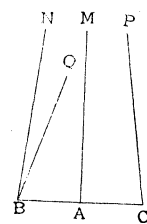∥ ≑ CP), and AM (in NBCP) bisects
⊥ the sect BC; then BN ∥ AM.

For if ray BN cut ray AM, also
ray CP would cut ray AM at the
same point (because MABN ≅
MACP), and this would be common
to the rays BN, CP themselves, al-

FIG. 8.

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* The third case being put before the other two, these can be
demonstrated together with more brevity and elegance, like case
2 of § 10.  [Author's note.]

though BN ‖ CP.   But every ray BQ (in CBN) cuts ray CP; and so ray BQ cuts also ray AM. Consequently BN ‖ AN.

§ 9.   If BN ‖ AM, and MAP ⊥ MAB, and the ∠, which NBD makes with NBA (on that side of MABN, where MAP is) is <rt.∠; then MAP and NBD intersect.
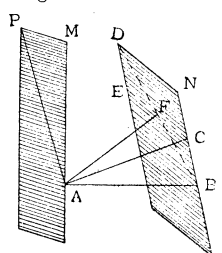
For let ∠BAM=rt.∠, and AC ⊥ BN (whether or not C falls on B), and CE ⊥ BN (in NBD); by hypothesis ∠ACE <rt.∠, and AF (⊥ CE) will fall in ACE.

Fig. 9.

Let ray AP be the intersection of the hemi-planes ABF, AMP (which have the point A common); since BAM ⊥ MAP, ∠BAP=∠BAM =rt.∠.

If finally the hemi-plane ABF is placed upon the hemi-plane ABM (A and B remaining), ray AP will fall on ray AM; and since AC ⊥ BN, and sect AF<sect AC, evidently sect AF will terminate within ray BN, and so BF falls in ABN.  But in *this* position, ray BF cuts ray AP (because BN ‖ AM);  and so ray AP and ray BF intersect also in *the original* position;  and the point of section is common to the  hemi-planes MAP and NBD.   Therefore the hemi-planes MAP and NBD intersect.   Hence follows eas-

[6]

ily that the hemi-planes MAP and NBD inter-
sect if the sum of the interior angles which
they make with MABN is <st.∠.
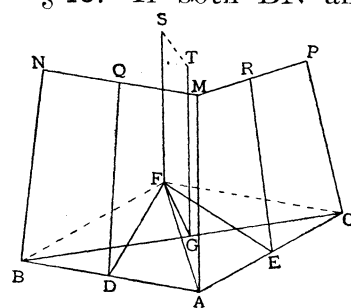
§ 10.  If both BN and CP ∥ ≏ AM;  also is
BN ∥ ≏ CP.



FIG. 10.

For either MAB
and MAC make an
angle, or they are in
a plane.

If the first; let the
hemi-plane QDF bi-
sect ⊥ sect AB; then
DQ ⊥ AB, and so DQ
∥ AM (§ 8); likewise if hemi-plane ERS bisects
⊥ sect AC, is ER ∥ AM;  whence (§ 7) DQ ∥ ER.

Hence follows easily (by § 9), the hemi-
planes QDF and ERS intersect, and have (§ 7)
their intersection FS ∥ DQ, and (on account of
BN ∥ DQ) also FS ∥ BN.  Moreover (for any
point of FS) FB=FA=FC, and the straight
FS falls in the plane TGF, bisecting ⊥ sect BC.
But (by § 7) (since FS ∥ BN) also GT ∥ BN.
In the same way is proved GT ∥ CP.  Mean-
while GT bisects ⊥ sect BC; and so TGBN≅
TGCP (§ 1), and BN ∥ ≏ CP.

If BN, AM and CP are in a plane, let (fall-
ing without this plane) FS ∥ ≏ AM; then (from

what precedes) FS ‖ ≏ both to BN and to CP, and so also BN ‖ ≏ CP.

§ 11.  Consider the aggregate of the point A, and *all* points of which any one B is such, that if BN ‖ AM, also BN ≏ AM; call it F; but the intersection of F with any plane containing the sect AM call L.

F has a point, and one only, on any straight ‖ AM; and evidently L is divided by ray AM into two congruent parts.

Call the ray AM *the axis* of L.  Evidently also, in any plane containing the sect AM, there is for the *axis* ray AM a single L.  Call any L of this sort the L of this ray AM (in the plane considered, being understood).  Evidently by revolving L around AM we describe the F of which ray AM is called the axis, and in turn F *may be ascribed to the axis ray AM.*

[7]    § 12.  If B is anywhere on the L of ray AM, and BN ‖ ≏ AM (§ 11); then the L of ray AM and the L of ray BN *coincide.*  For suppose, in distinction, L′ the L of ray BN.  Let C be anywhere in L′, and CP ‖ ≏ BN (§ 11).  Since BN ‖ ≏ AM, so CP ‖ ≏ AM (§ 10), and so C also will fall on L.  And if C is anywhere on L, and CP ‖ ≏ AM; then CP ‖ ≏ BN (§ 10); and C also falls on L′ (§ 11).  Thus L and L′ are the

same; and every ray BN is also axis of L, and between all axes of this L, is $\eqcirc$.

The same is evident in the same way of F.

§ 13.   If BN ∥ AM, and CP ∥ DQ, and∠BAM +∠ABN=st.∠;   then  also  ∠DCP+∠CDQ= st.∠.
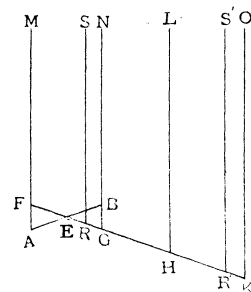


FIG. 11.

For  let  EA= EB,  and  EFM= DCP (§ 4).  Since ∠BAM+∠ABN =st. ∠=∠ABN+ ∠ABG,  we have ∠EBG=∠EAF; and so if also BG =AF, then△EBG ≅△EAF, ∠BEG=∠AEF  and  G  will  fall  on the ray FE.   Moreover ∠GFM+∠FGN=st.∠ (since ∠EGB=∠EFA).

Also GN ∥ FM (§ 6).

Therefore if  MFRS≅PCDQ, then RS ∥ GN (§ 7), and  R  falls  within  or  without  the sect FG (unless sect CD=sect FG, where the thing now is  evident).

I.  In the first case ∠FRS is not >(st.∠−∠ RFM=∠FGN), since  RS ∥ FM.   But as RS ∥ GN, also ∠FRS is not <∠FGN; and so ∠FRS =∠FGN,  a n d   ∠RFM+∠FRS=∠GFM+∠

FGN=st.∠.    Therefore also  ∠DCP+∠CDQ
=st.∠.

II.  If R falls without the sect FG; then
∠NGR=∠MFR,  and let  MFGN≅NGHL≅
LHKO, and so on, until FK=FR or begins to
be >FR.    Then KO ‖ HL ‖ FM (§7).

If K  falls  on  R, then  KO  falls  on  RS (§ 1);
and  so  ∠RFM+∠FRS=∠KFM+∠FKO=∠
KFM+∠FGN=st.∠;  but if R falls within the
sect  HK, then  (by I)  ∠RHL+∠KRS=st.∠=
∠RFM+∠FRS=∠DCP+∠CDQ.

§ 14.   If BN ‖ AM, and CP ‖ DQ, and ∠BAM
+∠ABN<st.∠;  then  also  ∠DCP+∠CDQ<
st.∠.

For if ∠DCP+∠CDQ were not <st.∠, and
so (by § 1)  were =st.∠,  then  (by § 13)  also ∠
BAM+∠ABN=st.∠ (contra hyp.).

§ 15.   Weighing §§ 13 and 14, *the System of
Geometry resting on the hypothesis of the
truth of Euclid's Axiom XI is called Σ; and
the system founded  on the contrary hypoth-*
[8] *esis is S.*

*All things which are not expressly said to
be in Σ or in S, it is understood  are enunci-
ated absolutely, that  is  are  asserted  true
whether Σ or S is reality.*

§ 16.   If AM is the axis of any L; then L, in ᴢ is a straight ⊥ AM.

For suppose BN an axis from any point B of L; in ᴢ, ∠BAM+∠ABN =st.∠, and so ∠BAM=rt.∠.

And if C is any point of the straight AB, and CP ‖ AM; then (by § 13) CP≏AM, and so C on L (§ 11).



FIG. 12.

But in S, no three points A, B, C on L or on F are in a straight.   For some one of the axes AM, BN, CP (e. g. AM) falls between the two others; and then (by § 14) ∠BAM and ∠CAM are each <rt.∠.

§ 17.   *L in S also is a line, and F a surface.*   For (by § 11) any plane ⊥ to the axis ray AM (through any point of F) cuts F in [the circumference of] a circle, of which the plane (by § 14) is ⊥ to no other axis ray BN. If we revolve F about BN, any point of F (by § 12) will remain on F, and the section of F with a plane not ⊥ ray BN will describe a surface; and whatever be the points A, B taken on it, F can so be congruent to itself that A falls upon B (by § 12); therefore F is *a uniform surface.*
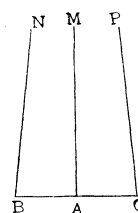
Hence evidently (by §§ 11 and 12) L is a uniform line.*

§ 18.   *The intersection* with F of *any plane*, drawn through a point A of F obliquely to the axis AM, is, in S, *a circle.*

For take A, B, C, three points of this section, and BN, CP, axes; AMBN and AMCP make an angle, for otherwise the plane determined by A, B, C (from § 16) would contain AM, (contra hyp.).   Therefore the planes bisecting ⊥ the sects AB, AC intersect (§ 10) in some *axis* ray FS (of F), and FB=FA=FC.
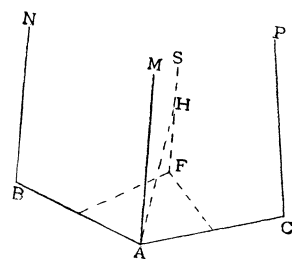
Make AH ⊥ FS, and revolve FAH about FS; A will describe a circle of radius HA, passing through B and C, and situated *both* in F and in [9] the plane ABC; nor have F and the plane ABC anything in common but ⊙ HA (§ 16).

FIG. 13.

It is also evident that in revolving the portion FA of the line L (as radius) in F around F, its extremity will describe ⊙ HA.

---

* It is not necessary to restrict the demonstration to the system S; since it may easily be so set forth. that it holds absolutely for S and for Σ.

§ 19.  *The perpendicular* BT to the axis BN of L (falling in the plane of L) is, in S, *tangent* to L.  For L has in ray BT no point except B (§ 14), but if BQ falls in TBN, then the center of the section of the plane through BQ perpendicular to TBN with the F of ray BN (§ 18) is evidently located on ray BQ; and if sect BQ is a diameter, evidently ray BQ cuts in Q the line L of ray BN.
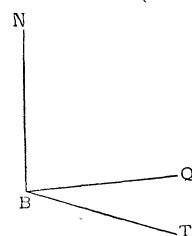
Fig. 14.

§ 20.  Any two points of F determine a line L (§§ 11 and 18); and since (from §§ 16 and 19) L is ⊥ to all its axes, every ∠ of lines L in F is equal to the ∠ of the planes drawn through its sides perpendicular to F.

§ 21.  Two L form lines, ray AP and ray BD, in the same F, making with a third L form AB, a sum of interior angles < st.∠, intersect.
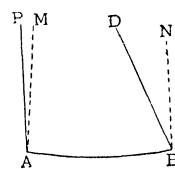
(By line AP in F, is to be understood the line L drawn through A and P, but by ray AP that half of this line beginning at A, in which P falls.)

Fig. 15.

For if AM, BN are axes of F, then the hemiplanes AMP, BND intersect (§ 9); and F cuts

their intersection (by §§ 7 and 11); and so also ray AP and ray BD intersect.

From this it is evident that Euclid's Axiom *XI* and all things which are claimed in geometry and plane trigonometry hold good *absolutely* in F, L lines being substituted in place of straights: therefore the trigonometric functions are taken here in the same sense as in Σ; and the circle of which the L form radius $= r$ in F, is $= 2\pi r$; and likewise area of $\odot r$ (in F) $= \pi r^2$ (by $\pi$ understanding ½⊙1 in F, or the known 3.1415926...)

§ 22.  If ray AB were the L of ray AM, and C on ray AM; and the ∠CAB (formed by the straight ray AM and the L form line ray AB), carried first along [10] the ray AB, then along the ray BA, always forward to infinity: the path CD of C will be the line L of CM.
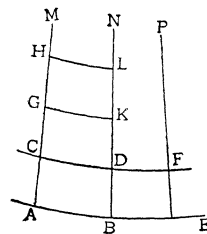


FIG. 16.

For let D be any point in line CD (called later L′), let DN be ∥ CM, and B the point of L falling on the straight DN. We shall have BN ≏ AM, and sect AC = sect BD, and so DN ≏ CM, consequently D in L′. But if D in L′ and DN ∥ CM, and B the point of L on the straight DN; we shall have AM ≏ BN and CM ≏ DN, whence manifestly sect BD = sect AC,

and D will fall on the path of the point C, and L′ and the line CD are the same.  Such an L′ is designated by L′∥L.

§ **23.**  If the L form line CDF∥ABE (§ 22), and AB=BE, and the rays AM, BN, EP are axes; manifestly CD=DF; and if any three points A, B, E are of line AB, and AB=n.CD, we shall also have AE=n.CF; and so (manifestly even for AB, AE, DC incommensurable), AB:CD=AE:CF, and AB:CD is *independent of AB, and completely determined by AC.*

This ratio AB:CD is designated by the capital letter (as X) corresponding to the small letter (as $x$) by which we represent the sect AC.

§ **24.**  Whatever be $x$ and $y$; (§ 23), $Y = X^{\frac{y}{x}}$.

For, one of the quantities $x, y$ is a multiple of the the other (e. g. $y$ of $x$), or it is not.

If $y = n.x$, take $x = AC = CG = GH = \&c.$, until we get $AH = y$.

Moreover, take CD∥GK∥HL.

We have ((§ 23) X=AB:CD=CD:GK=GK: HL; and so

$$\frac{AB}{HL} = \left[\frac{AB}{CD}\right]^{n}$$

or $Y = X^{n} = X^{\frac{y}{x}}$.

If $x, y$ are multiples of $i$, suppose $x = mi$, and $y = ni$; (by the preceding) $X = I^{m}$, $Y = I^{n}$, consequently

$$Y = X^{\frac{n}{m}} = X^{\frac{y}{x}}$$

The same is easily extended to the case of the incommensurability of $x$ and $y$.

But if q=y−x, manifestly Q=Y:X.

It is also manifest that in $\Sigma$, for any $x$, we have X=1, but in S is X>1, and for any AB [11] and ABE there is such a CDF ‖ AB, that CDF =AB, whence AMBN≅AMEP, though the first be any multiple of the second; which indeed is singular, but evidently does not prove the absurdity of S.

§ 25. *In any rectilineal triangle, the circles with radii equal to its sides are as the sines of the opposite angles.*
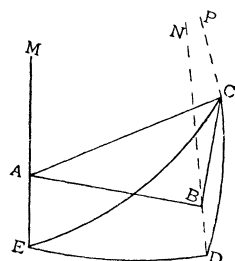


FIG. 17.

For take ∠ABC=rt.∠, and AM⊥BAC, and BN and CP ‖ AM; we shall have CAB ⊥ AMBN, and so (since CB⊥ BA), CB⊥AMBN, consequently CPBN ⊥ AMBN.

Suppose the F of ray CP cuts the straights BN, AM respectively in D and E, and the bands CPBN, CPAM, BNAM along the L form lines CD, CE, DE. Then (§ 20) ∠CDE= the angle of NDC, NDE, and so =rt.∠; and by like reasoning ∠CED=∠CAB. But (by § 21) in the L line △ CDE (supposing always here the radius =1),

EC:DC=1:sin DEC=1:sin CAB.

Also (by § 21)

EC:DC=⊙EC:⊙DC (in F)=⊙AC:⊙BC (§ 18);

and so is also

⊙AC:⊙BC=1:sin CAB;

whence the theorem is evident for any triangle.

§ 26.  *In any spherical triangle, the sines of the sides are as the sines of the angles opposite.*
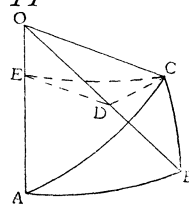
For take ∠ABC=rt.∠, and CED ⊥ to the radius OA of the sphere.  We shall have CED ⊥ AOB, and (since also BOC ⊥ BOA),  CD ⊥ OB.  But in the triangles CEO, CDO (by § 25) ⊙EC:⊙OC:⊙DC=sin COE : 1 : sin COD=sin AC : 1 : sin BC; meanwhile also (§ 25) ⊙EC : ⊙DC=sin CDE : sin CED.  Therefore, sin AC : sin BC=sin CDE : sin CED; but CDE= rt.∠=CBA, and CED=CAB.  Consequently

sin AC : sin BC=1 : sin A.

*Spherical trigonometry, flowing from this, is thus established independently of Axiom XI.*

§ 27.  If AC and BD are ⊥ AB, and CAB is carried along the straight AB; we shall have, designating by CD the path of the point C,

CD : AB=sin *u* : sin *v*.

Fig. 19.

For take DE ⊥ CA; in the triangles ADE, ADB (by § 25)

$$\odot ED : \odot AD : \odot AB = \sin u : 1 : \sin v.$$

In revolving BACD about AC, B describes ⊙AB, and D describes ⊙ED; and designate here by s⊙CD the path of the said CD. Moreover, let there be any [12] polygon BFG... inscribed in ⊙AB.

Passing through all the sides BF, FG, &c., planes ⊥ to ⊙AB we form also a polygonal figure of the same number of sides in s⊙CD, and we may demonstrate, as in § 23, that CD : AB =DH : BF=HK : FG, &c., and so

DH+HK &c. : BF+FG &c. : =CD : AB.

If each of the sides BF, FG... approaches the limit zero, manifestly

$$BF+FG+\ldots \doteq \odot AB \quad \text{and}$$
$$DH+HK+\ldots \doteq \odot ED.$$

Therefore also ⊙ED : ⊙AB=CD : AB. But we had ⊙ED : ⊙AB=sin $u$ : sin $v$. Consequently

$$CD : AB = \sin u : \sin v.$$

If AC goes away from BD to infinity, CD : AB, and so also sin $u$ : sin $v$ remains *constant;* but $u \doteq$ rt. ∠ (§ 1), and if DM ‖ BN, $v \doteq z$; whence CD : AB=1 : sin $z$.

The path called CD will be denoted by CD ‖ AB.

§ **28.** If $BN \parallel \eqsim AM$, and C in ray AM, and $AC=x$: we shall have (§ 23)
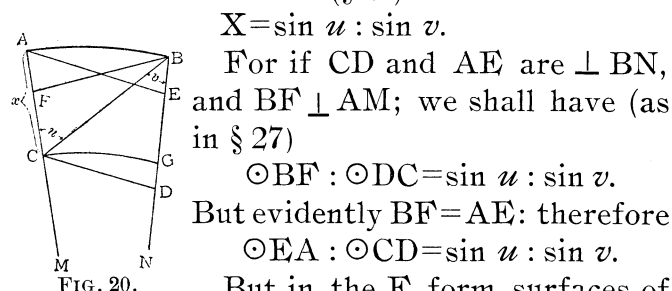
$$X=\sin u : \sin v.$$

For if CD and AE are $\perp$ BN, and $BF \perp AM$; we shall have (as in § 27)

$$\odot BF : \odot DC = \sin u : \sin v.$$

But evidently BF=AE: therefore

$$\odot EA : \odot CD = \sin u : \sin v.$$

But in the F form surfaces of AM and CM (cutting AMBN in AB and CG) (by § 21)

$$\odot EA : \odot DC = AB : CG = X.$$

Therefore also

$$X = \sin u : \sin v.$$

FIG. 20.

§ **29.** If $\angle BAM = \mathrm{rt}.\angle$, and sect AB=$y$, and BN ‖ AM, we shall have in S
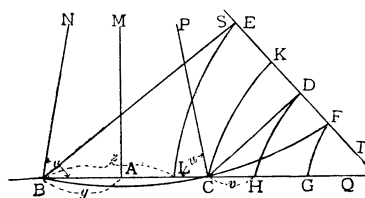
$$Y = \cotan \tfrac{1}{2} u.$$

For, if sect AB= sect AC, and CP ‖ AM (and so $BN \parallel \eqsim CP$), and $\angle PCD = \angle QCD$; there is given (§ 19) DS $\perp$ ray CD, so that DS ‖ CP, and so (§ 1) DT ‖ CQ. Moreover, if BE $\perp$ ray DS, then (§ 7) DS ‖ BN, and so (§ 6)

FIG. 21.

BN ‖ ES, and (since DT ‖ CG) BQ ‖ ET; consequently (§ 1) ∠EBN=∠EBQ.  Let BCF be an L-line of BN, and FG, DH, CK, EL, L form lines of FT, DT, CQ and ET; evidently (§ 22) HG=DF=DK=HC; therefore,

$$CG=2CH=2v.$$

Likewise it is evident BG=2BL=2$z$.

But BC=BG−CG; wherefore $y=z-v$, and so (§ 24) Y=Z : V.

Finally (§ 28)

$$Z=1 : \sin \tfrac{1}{2}\, u,$$
$$\text{and } V=1 : \sin (\text{rt.}\angle - \tfrac{1}{2}\, u),$$

consequently  Y=cotan $\tfrac{1}{2}\, u$.

§ 30.  However, it is easy to see (by § 25) [13] that the solution of the problem of Plane



FIG. 22.

Trigonometry, in S, requires the expression of the circle in terms of the radius; but this can by obtained by the rectification of L.
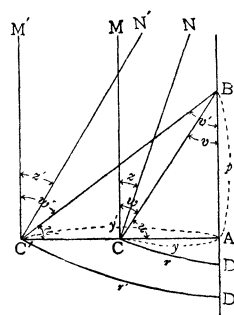
Let AB, CM, C′M′ be ⊥ ray AC, and B anywhere in ray AB; we shall have (§ 25)

$$\sin u : \sin v = \odot p : \odot y,$$
$$\text{and } \sin u' : \sin v' = \odot p' : \odot y';$$

and so $\dfrac{\sin u}{\sin v}.\odot y = \dfrac{\sin u'}{\sin v'}.\odot y'.$

But (by § 27) $\sin v : \sin v' = \cos u : \cos u'$; consequently $\dfrac{\sin u}{\cos u}.\odot y = \dfrac{\sin u'}{\cos u'}.\odot y'$;

or   $\odot y : \odot y' = \tan u' : \tan u = \tan w : \tan w'$.

Moreover, take CN and C′N′ ∥ AB, and CD, C′D′ L-form lines ⊥ straight AB;  we shall have also (§21)

$$\odot y : \odot y' = r : r', \text{ and so}$$
$$r : r' = \tan w : \tan w'.$$

Now let p beginning  from A increase to infinity;  then $w \doteq z$, and $w' \doteq z'$, whence also $r : r' = \tan z : \tan z'$.

Designate by $i$ the *constant*
$$r : \tan z \ (independent \text{ of } r);$$
whilst   $y \doteq 0$,
$$\frac{r}{y} = \frac{i \tan z}{y} \doteq 1, \text{ and so}$$

$\dfrac{y}{\tan z} \doteq i$.   From § 29, $\tan z = \tfrac{1}{2}\,(Y - Y^{-1})$;

therefore      $\dfrac{2y}{Y - Y^{-1}} \doteq i,$

or (§ 24)      $\dfrac{2y.\mathrm{I}^{\frac{y}{i}}}{\mathrm{I}^{\frac{2y}{i}} - 1} \doteq i.$

But we know  the limit of this expression (where $y \doteq 0$) is

$$\frac{i}{\text{nat. log I}}.\quad \text{Therefore}$$

$$\frac{i}{\text{nat. } \log \text{I}} = i, \text{ and}$$

$$\text{I} = e = 2.7182818\ldots,$$

which noted quantity shines forth here also.

If obviously henceforth $i$ denote that sect of which the $\text{I}=e$, we shall have

$$r = i \tan z.$$

But (§ 21) $\odot y = 2\pi r$; therefore

$$\odot y = 2\pi i \tan z = \pi i \ (\text{Y} - \text{Y}^{-1}) = \pi i \left[ e^{\frac{y}{i}} - e^{\frac{-y}{i}} \right]$$

$$= \frac{\pi y}{\text{nat. } \log \text{Y}} (\text{Y} - \text{Y}^{-1}) \text{ (by § 24).}$$

§ **31.** For the trigonometric solution of all right-angled rectilineal *triangles* (whence the resolution of all *triangles* is easy , in S, three [14] equations suffice : indeed ($a$, $b$ denoting the sides, $c$ the hypothenuse, and $\alpha$, $\beta$ the angles opposite the sides) an equation expressing the relation

1st, between $a$, $c$, $\alpha$;

2d, between $a$, $\alpha$, $\beta$;

3d, between $a$, $b$, $c$;

of course from these equations emerge three others by elimination.



FIG. 23.

From §§ 25 and 30
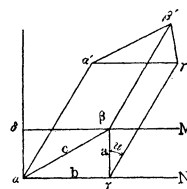
$$1 : \sin \alpha = (\text{C} - \text{C}^{-1}) : (\text{A} - \text{A}^{-1}) =$$

$$= \left[ e^{\frac{c}{i}} - e^{\frac{-c}{i}} \right] : \left[ e^{\frac{a}{i}} - e^{\frac{-a}{i}} \right] \text{ (equation for } c, a \text{ and } \alpha).$$

II.   From § 27 follows (if $\beta M \parallel \gamma N$)
$\cos a : \sin \beta = 1 : \sin u$; but from § 29
$$1 : \sin u = \tfrac{1}{2}(A+A^{-1});$$
therefore  $\cos a : \sin \beta = \tfrac{1}{2}(A+A^{-1}) = \tfrac{1}{2}\left[e^{\frac{a}{i}} + e^{\frac{-a}{i}}\right]$
(equation for $a$, $\beta$ and $a$).

III.   If $aa' \perp \beta a\gamma$, and $\beta\beta'$ and $\dot\gamma\gamma' \parallel aa'$ (§ 27),
and $\beta'a'\gamma' \perp aa'$; manifestly (as in § 27)
$$\frac{\beta\beta'}{\gamma\gamma'} = \frac{1}{\sin u} = \tfrac{1}{2}(A+A^{-1});$$

$$\frac{\gamma\gamma'}{aa'} = \tfrac{1}{2}(B+B^{-1}\,;$$

and $\dfrac{\beta\beta'}{aa'} = \tfrac{1}{2}(C+C^{-1})$;  consequently

$$\tfrac{1}{2}(C+C^{-1}) = \tfrac{1}{2}(A+A^{-1}) \cdot \tfrac{1}{2}(B+B^{-1}), \text{ or}$$

$$\left[e^{\frac{c}{i}} + e^{\frac{-c}{i}}\right] = \tfrac{1}{2}\left[e^{\frac{a}{i}} + e^{\frac{-a}{i}}\right]\left[e^{\frac{b}{i}} + e^{\frac{-b}{i}}\right]$$

(equation for $a$, $b$ and $c$).

If $\gamma a\delta = \text{rt.} \angle$, and $\beta\,\delta \perp a\delta$;
$$\odot c : \odot a = 1 : \sin a, \text{ and}$$
$$\odot c : \odot(d = \beta\delta) = 1 : \cos a,$$
and so (denoting by $\odot x^2$, for any $x$, the product
$\odot x.\odot x$) manifestly
$$\odot a^2 + \odot d^2 = \odot c^2.$$

But (by § 27 and II)
$\odot d = \odot b.\tfrac{1}{2}(A+A^{-1})$, consequently
$$\left[e^{\frac{c}{i}} - e^{\frac{-c}{i}}\right]^2 = \tfrac{1}{4}\left[e^{\frac{a}{i}} + e^{\frac{-a}{i}}\right]^2 \cdot \left[e^{\frac{b}{i}} - e^{\frac{-b}{i}}\right]^2 + \left[e^{\frac{a}{i}} - e^{\frac{-a}{i}}\right]^2$$

*another* equation for $a$, $b$ and $c$ (the second

member of which may be easily reduced to a form *symmetric* or *invariable*).    [15]

Finally, from

$\dfrac{\cos a}{\sin \beta} = \tfrac{1}{2}(A+A^{-1})$, and $\dfrac{\cos \beta}{\sin a} = \tfrac{1}{2}(B+B^{-1})$, we get (by III)

$$\cot a \cot \beta = \tfrac{1}{2} \left( e^{\frac{c}{i}} + e^{\frac{-c}{i}} \right)$$

(equation for $a$, $\beta$, and $c$.

§ **32.** It still remains to show briefly the mode of resolving *problems* in S, which being accomplished (through the more obvious examples), finally will be candidly said what this theory shows.

I.   Take AB a line in a plane, and $y = f(x)$
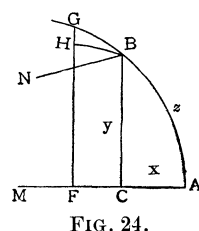


FIG. 24.

its equation in rectangular coordinates, call $dz$ any increment of $z$, and respectively $dx, dy, du$ the increments of $x$, of $y$, and of the area $u$, corresponding to this $dz$; take BH ‖ CF, and express (from § 31) $\dfrac{\text{BH}}{dx}$ by means of $y$, and seek

the *limit* of $\dfrac{dy}{dx}$ when $dx$ tends towards the limit zero (which is understood where a limit of this sort is sought): then will become known

also the limit of $\dfrac{dy}{\text{BH}}$, and so $\tan$ HBG; and

(since HBC manifestly is neither $>$ nor $<$, and so $=$rt.$\angle$), the *tangent* at B of BG will be determined by $y$.

II.   It can be demonstrated

$$\frac{dz^2}{dy^2 + \overline{BH}^2} \doteq 1.$$

Hence is found the *limit* of $\frac{dz}{dx}$, and thence, by integration, $z$ (expressed in terms of $x$.

And of any line *given in the concrete,* the equation in S can be found; e. g., of L.   For if ray AM be the axis of L; then any ray CB from ray AM cuts L [since (by § 19) any straight from A except the straight AM will cut L]; but (if BN is axis)

$$X = 1 : \sin \ CBN \ (\S\, 28),$$
and $Y = \text{cotan } \tfrac{1}{2} \ CBN \ (\S\, 29),$ whence
$$Y = X + \sqrt{X^2 - 1}.$$

or
$$e^{\frac{y}{i}} = e^{\frac{x}{i}} + \sqrt{e^{\frac{2x}{i}} - 1}, \qquad [16]$$

the equation sought.

Hence we get

$$\frac{dy}{dx} \doteq X(X^2 - 1)^{-\frac{1}{2}};$$

and $\dfrac{BH}{dx} \doteq 1 : \sin \ CBN = X;$  and so

$$\frac{dy}{BH} \doteq (X^2 - 1)^{-\frac{1}{2}};$$

$$1+\frac{dy^2}{\mathrm{BH}^2}=\mathrm{X}^2(\mathrm{X}^2-1)^{-1},$$

$$\frac{dz^2}{\mathrm{BH}^2}=\mathrm{X}^2(\mathrm{X}^2-1)^{-1},$$

and $\dfrac{dz}{\mathrm{BH}}=\mathrm{X}(\mathrm{X}^2-1)^{-\frac{1}{2}}$, and

$\dfrac{dz}{dx}=\mathrm{X}^2(\mathrm{X}^2-1)^{-\frac{1}{2}}$, whence, by inte-

gration, we get (as in § 30)
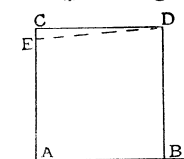
$$z=i(\mathrm{X}^2-1)^{\frac{1}{2}}=i \cot \mathrm{CBN}.$$

III.   Manfestly

$$\frac{du}{dx}=\frac{\mathrm{HFCBH}}{dx},$$

which (unless given in $y$) now first is to be ex-
pressed in terms of $y$; whence we get $u$ by
integrating.


Fig. 25.

If $\mathrm{AB}=p$, $\mathrm{AC}=q$, $\mathrm{CD}=r$, and
$\mathrm{CABDC}=s$; we might show (as
in II) that
$\dfrac{ds}{dq}=r$, which $=\frac{1}{2}p\left[e^{\frac{q}{i}}-e^{\frac{-q}{i}}\right]$,

and, integrating, $s=\frac{1}{2}pi\left[e^{\frac{q}{i}}-e^{\frac{-q}{i}}\right]$

This can also be deduced apart from inte-
gration.

For example, the equation of the circle (from
§ 31, III), of the straight (from § 31, II), of a
conic (by what precedes), being expressed, the

areas bounded by these lines could also be expressed.

We know, that a surface $t$, ||| to a plane figure $p$ (at the distance $q$), is to $p$ in the ratio of the second powers of homologous lines, or as

$$\tfrac{1}{4}\left(e^{\frac{q}{i}}-e^{\frac{-q}{i}}\right)^2 : 1.$$

It is easy to see, moreover, that the calculation of volume, treated in the same manner, requires two integrations (since the differential itself here is determined only by integration); and before all must be investigated the [17] volume contained between $p$ and $t$, and the aggregate of all the straights $\perp p$ and joining the boundaries of $p$ and $t$.

We find for the volume of this solid (whether by integration or without it)

$$\tfrac{1}{8}pi\left(e^{\frac{2q}{i}}-e^{\frac{-2q}{i}}\right)+\tfrac{1}{2}pq.$$

The surfaces of bodies may also be determined in S, as well as the *curvatures,* the *involutes*, and *evolutes* of any lines, etc.

As to curvature; this in S either is the curvature of L, or is determined either by the radius of a circle, or by the *distance* to a straight from the curve ||| to this straight; since from what precedes, it may easily be shown, that in a plane there are no uniform lines other than L-lines, circles and curves ||| to a straight.

IV.  For the circle (as in III) $\dfrac{d \text{ area } \odot x}{dx}=$
$\odot x$, whence (by § 29), integrating,

$$\text{area } \odot x = \pi i^2 \left[ e^{\frac{x}{i}} - 2 + e^{\frac{-x}{i}} \right].$$

V.  For the area CABDC=$u$ (inclosed by an L form line AB=$r$, the ‖ to this, CD=$y$, and the sects AC=BD=$x$) $\dfrac{du}{dx}=y;$ and (§ 24) $y=re^{\frac{-x}{i}}$, and so, (integrating) $u=ri\left[ 1-e^{\frac{-x}{i}} \right].$

If $x$ increases to infinity, then, in S, $e^{\frac{-x}{i}}=0$, and so $u=ri$.  By the *size* of MABN, in future this limit is understood.

In like manner is found, if $p$ is a figure on F, the space included by $p$ and the aggregate of axes drawn from the boundaries of $p$ is equal to ½$pi$.
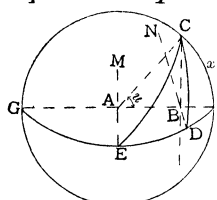
FIG. 26.

VI.  If the angle at the center of a segment $z$ of a sphere is $2u$, and a great circle is $p$, and $x$ the arc FC (of the angle $u$); (§ 25)

$$1 : \sin u = p : \odot BC,$$

and hence $\odot BC = p \sin u.$    [18]

FIG. 27.

Meanwhile $x=\dfrac{pu}{2\pi}$, and $dx=\dfrac{pdu}{2\pi}.$

Moreover, $\frac{dz}{dx}=\odot BC$, and hence

$$\frac{dz}{du}=\frac{p^2}{2\pi}\sin u,\text{ whence (integrating)}$$

$$z=\frac{\text{ver sin } u}{2\pi}p^2.$$

The F may be conceived on which P falls (passing through the middle F of the segment); through AF and AC the planes FEM, CEM are placed, perpendicular to F and cutting F along FEG and CE; and consider the L form CD (from C ⊥ to FEG), and the L form CF; (§ 20) CEF=$u$, and (§ 21)

$$\frac{FD}{p}=\frac{\text{ver sin } u}{2\pi},\text{ and so } z=FD.p.$$

But (§ 21) $p=\pi.FGD$; therefore

$$z=\pi.FD.FDG. \quad \text{But (§ 21)}$$



FD.FDG=FC.FC; consequently
$z=\pi.FC.FC=\text{area }\odot FC$, in F.

Now let BJ=CJ=$r$; (§ 30)
$2r=i(Y—Y^{-1})$, and so (§ 21)
area $\odot 2r$ (in F) $=\pi i^2(Y-Y^{-1})^2$.

FIG. 28.        Also (*IV*)

$$\text{area }\odot 2y=\pi i^2(Y^2—2+Y^{-2});$$

therefore, area $\odot 2r$ (in F) =area $\odot 2y$, and so *the surface z of a segment of a sphere is equal to the surface of the circle described with the chord* FC *as a radius.*
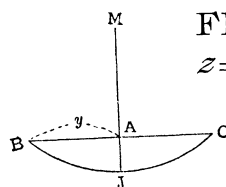
Hence the whole surface of the sphere

$$=\text{area}\odot\text{FG}=\text{FDG}.p=\frac{p^2}{\pi},$$

*and the surfaces of spheres are to each other as the second powers of their great circles.*

VII. In like manner, in S, the volume of the sphere of radius $x$ is found

$$=\tfrac{1}{2}\pi i^3(X^2-X^{-2})-2\pi i^2 x;$$
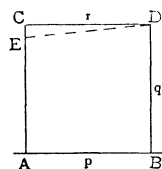
the surface generated by the revolution of the line CD about AB

$$=\tfrac{1}{2}\pi i p(Q^2-Q^{-2}),$$

and the body described by CABDC

$$=\tfrac{1}{4}\pi i^2 p(Q-Q^{-1})^2.$$



FIG. 29.

*But in what manner all things treated from (IV) even to here, also may be reached apart from integration, for the sake of brevity is suppressed.*

It can be demonstrated that *the limit of every expression containing the letter i* (and so resting upon the hypothesis that $i$ is given), [19] *when i increases to infinity, expresses the quantity simply for* Σ (and so for the hypothesis of no $i$), if indeed the equations do not become identical.

But beware lest you understand to be supposed, that the system itself may be varied (for it is entirely determined in itself and by itself); but only *the hypothesis,* which may be

done successively, as long as we are not conducted to an absurdity. *Supposing* therefore that, in *such* an expression, the letter $i$, in case S is reality, designates that unique quantity whose I$=e$; but if $\Sigma$ is actual, the said limit is supposed to be taken in place of the expression: manifestly *all the expressions originating from the hypothesis of the reality of S* (*in this sense*) *will be true absolutely, although it be completely unknown whether or not $\Sigma$ is reality*

So e. g. from the expression obtained in § 30 easily (and as well by aid of differentiation as apart from it) emerges the known value in $\Sigma$,

$$\odot x = 2\pi x;$$

from I (§ 31) suitably treated, follows

$$1 : \sin a = c : a;$$

but from II

$$\frac{\cos a}{\sin \beta} = 1, \text{ and so}$$

$$a + \beta = \text{rt.} \angle;$$

the first equation in III becomes identical, and so is true in $\Sigma$, although it there determines nothing; but from the second follows

$$c^2 = a^2 + b^2.$$

*These are the known fundamental equations of plane trigonometry in $\Sigma$.*

Moreover, we find (from § 32) in $\Sigma$, the area and the volume in III each $=pq;$ from IV

$$\text{area} \odot x = \pi x^2;$$

(from VII) the globe of radius $x$

$$= \tfrac{4}{3}\pi x^3, \text{ etc.}$$

The theorems enunciated at the end of VI are manifestly *true unconditionally*.

§ 33.  It still remains to set forth (as promised in § 32) what this theory means.

I.  Whether $\Sigma$ or some one S is reality, remains undecided.

II.  All things deduced from the hypothesis of the falsity of Axiom *XI* (always to be understood in the sense of § 32) are *absolutely true,* and so in this sense, *depend upon no hypothesis*.

There is therefore *a plane trigonometry a priori, in which the system alone really re-* [20] *mains unknown;* and so where remain unknown solely the *absolute* magnitudes in the expressions, but where a *single* known case would manifestly fix the whole system.  But spherical trigonometry is established absolutely in § 26.

(And we have, on F, a geometry wholly analogous to the plane geometry of $\Sigma$.)

III.  If it were agreed that $\Sigma$ exists, nothing more would be unknown in this respect; but

if it were *established* that $\Sigma$ *does not exist*, then (§ 31), (e. g.) from the sides *x, y,* and the rectilineal angle they include being given in a special case, manifestly it would be impossible in itself and by itself to solve absolutely the triangle, that is, to determine *a priori* the other angles and *the ratio of the third side* to the two given; unless X, Y were determined, for which it would be necessary to have in concrete form a certain sect *a* whose A was known; and then *i* would be *the natural unit for length* (just as *e* is the base of *natural* logarithms).

If the existence of this *i* is determined, it will be evident how it could be constructed, at least very exactly, for practical use.

IV.    In the sense explained (I and II), it is evident that all things in space can be solved by the modern analytic method (within just limits strongly to be praised).

V.    Finally, to friendly readers will not be unacceptable; that for that case wherein not $\Sigma$ but S is reality, a rectilineal figure is constructed equivalent to a circle.

§ 34.    Through D we may draw DM ‖ AN in the following manner.    From D drop DB⊥AN; from any point A of the straight AB erect AC ⊥ AN (in DBA), and let fall DC⊥AC.    We

will have (§ 27)  ⊙CD : ⊙AB=1 : sin *z*, pro-
vided that DM ‖ BN.  But sin
*z* is not >1;  and so  AB is
not >DC.  Therefore a quad-
rant described from the cen-
ter A in BAC, with a radius
=DC, will have a point B or O in common with
ray BD.   In the first case, manifestly *z*=rt.∠;
but in the second case (§ 25)

　　　(⊙AO=⊙CD) : ⊙AB=1 : sin AOB,

and so *z*=AOB.

If therefore we take *z*=AOB, then DM will
be ‖ BN.

§ **35.**   If S were reality; we may, as follows,
draw a straight ⊥ to one arm of an acute angle, [21]
which is ‖ to the other.
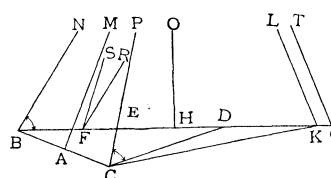
Take AM⊥BC, and
suppose  AB=BC  so
small (by § 19),  that
if  we  draw  BN ‖ AM
(§  34),   ABN > t h e
given angle.

Moreover  draw  CP ‖ AM  (§ 34);  and  take
NBG and PCD each equal to the given  angle;
rays BG and CD will cut; for if ray BG (fall-
ing *by construction* within NBC) cuts ray CP
in E; we shall have (since BN ≏ CP), ∠EBC<
∠ECB, and so EC<EB.  Take EF=EC, EFR

=ECD, and FS ∥ EP; then FS will fall within BFR.    For  since  BN ∥ CP, and  so  BN ∥ EP, and BN ∥ FS;  we shall have (§ 14)

∠FBN+∠BFS<(st.∠=FBN+BFR);

therefore, BFS<BFR.   Consequently, ray FR cuts ray EP, and so  ray CD also cuts ray EG in  some  point  D.    Take  now  DG=DC  and DGT=DCP=GBN;  we shall have (since CD ≏ GD) BN ≏ GT ≏ CP.    Let K (§ 19) be the point of the L-form line of BN falling in the ray BG, and KL  the  axis;  we  shall  have  BN ≏ KL, and so  BKL=BGT=DCP;  but also  KL ≏ CP: therefore manifestly K fall on G, and GT ∥ BN. But  if  HO  bisects ⊥ BG,  we  shall  have con-structed HO ∥ BN.

§ 36.  Having  given  the  ray  CP  and  the plane  MAB, take  CB ⊥ the plane  MAB,  BN  (in  plane BCP) ⊥ BC,  a n d  CQ ∥ BN (§ 34);  the intersection of ray CP  (if  this  ray  falls  within BCQ)  with  ray  BN  (in  the plane  CBN), and  so with  the plane MAB is found.    And  if  we  are  given the two planes PCQ, MAB, and  we have CB ⊥ to  plane  MAB,  CR ⊥ plane  PCQ; and  (in plane  BCR)  BN ⊥ BC, CS ⊥ CR,  BN  will fall in plane MAB, and CS in plane PCQ;  and the
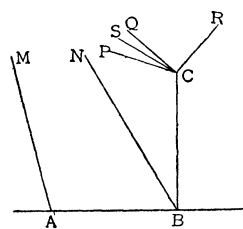
FIG. 32.

intersection of the straight BN with the straight CS (if there is one) having been found, the perpendicular drawn through this intersection, in PCQ, to the straight CS will manifestly be the intersection of plane MAB and plane PCQ.

§ 37. On the straight AM ‖ BN, is found such an A, that AM ≏ BN. If (by [22] § 34) we construct outside of the plane NBM, GT ‖ BN, and make BG⊥GT, GC=GB, and CP‖GT; and so place the hemiplane TGD that it makes
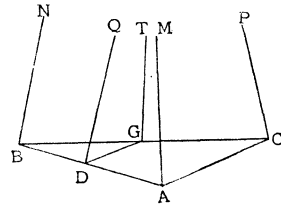


FIG. 33.

with hemi-plane TGB an angle equal to that which hemi-plane PCA makes with hemi-plane PCB; and is sought (by § 36) the intersection straight DQ of hemi-plane TGD with hemi-plane NBD; and BA is made ⊥DQ.

We shall have indeed, on account of the similitude of the triangles of L lines produced on the F of BN (§ 21), manifestly DB=DA, and AM≏BN.

Hence easily appears (L-lines being given by their extremities alone) we may also find a fourth proportional, or a mean proportional, and execute in this way in F, apart from Axiom XI, all the geometric constructions made

on the plane in $\Sigma$.   Thus e. g. a perigon can be geometrically divided into any special number of equal parts, if it is permitted to make this special partition in $\Sigma$.

§ **38.**   If we construct (by § 37) for example, NBQ=⅓ rt.$\angle$, and make (by § 35), in S, AM$\perp$ray BQ and ‖ BN, and determine (by § 37) IM$\doteq$BN; we shall have, if IA $=x$, (§ 28), X=1 : sin ⅓ rt. $\angle$=2, and $x$ will be constructed *geometrically*.

FIG. 34.

And NBQ may be so computed, that IA differs from $i$ less than by anything given, which happens for sin NBQ=$^1/e$.

§ **39.**   If (in a plane) PQ and ST are ‖‖‖ to the straight MN (§ 27), and AB, CD are equal perpendiculars to MN; manifestly $\triangle$DEC$\cong$ $\triangle$BEA; and so the angles (perhaps mixtilinear) ECP, EAT will fit, and EC=EA. If, moreover, CF=AG, then $\triangle$ACF$\cong$$\triangle$CAG, and each is half of the *quadrilateral* FAGC.

FIG. 35.

If FAGC, HAGK are two quadrilaterals of this sort on AG, between PQ and ST; their equivalence (as in Euclid) is evident, as also

the equivalence of the triangles AGC, AGH, standing on the same AG, and having their vertices on the line PQ. Moreover, ACF= CAG, GCQ=CGA, and ACF+ACG+GCQ= st.∠ (§ 32); and so also CAG+ACG+CGA= [23] st.∠; therefore, in any triangle ACG of this sort, the sum of the three angles =st.∠. But whether the straight AG may have fallen upon AG (which ⫴ MN), or not; *the equivalence* of the rectilineal triangles AGC, AGH, as well of themselves, *as of the sums of their angles,* is evident.

§ 40. *Equivalent triangles* ABC, ABD, (henceforth rectilineal), *hav-*


FIG. 36.

*ing one side equal, have the sums of their angles equal.*

For let MN bisect AC and BC, and take (through C) PQ⫴MN; the point D will fall on line PQ.

For, if ray BD cuts the straight MN in the point E, and so (§ 39) the line PQ at the distance EF=EB; we shall have △ABC=△ABF, and so also △ABD=△ABF, whence D falls at F.

But if ray BD has not cut the straight MN, let C be the point, where the perpendicular bisecting the straight AB cuts the line PQ, and

let GS=HT, so, that the line ST meets the
ray BD prolonged in a certain K (which it is
evident can be made in a way like as in § 4);
moreover take SR=SA, RO∥ST, and O the
intersection of ray BK with RO; then △ABR
=△ABO (§ 39), and so △ABC>△ABD (con-
tra hyp.).

§ **41**. *Equivalent triangles ABC, DEF
have the sums of their triangles equal.*



FIG. 37.

For let MN bisect
AC and BC, and PQ
bisect DF and FE;
and take RS∥MN,
and TO∥PQ; the per-
pendicular AG to RS
will equal the perpendicular DH to TO, or one
for example DH will be the greater.

In each case, the ⊙DF, from center A, has
with line-ray GS some point K in common,
and (§ 39) △ABK=△ABC=△DEF. But the
△AKB (by § 40) has the same angle-sum as
△DFE, and (by § 39) as △ABC. Therefore
also the triangles ABC, DEF have each the
same angle-sum.

In S the inverse of this theorem is true.

For take ABC, DEF two triangles having
equal angle-sums, and △BAL=△DEF; these
will have (by what precedes) equal angle-sums,

and so also will $\triangle$ABC and $\triangle$ABL, and hence manifestly

$$BCL+BLC+CBL=\text{st.}\angle.$$

However (by § 31), the angle-sum of any tri- [24] angle, in S, is $<\text{st.}\angle.$

Therefore L falls on C.

§ **42.**   Let $u$ be the supplement of the angle-sum of the $\triangle$ABC, but $v$ of $\triangle$DEF; then is $\triangle$ABC:$\triangle$DEF$=u:v$.


FIG. 38.

For if $p$ be the area of each of the triangles ACG, GCH, HCB, DFK, KFE; and $\triangle$ABC$=m.p$, and $\triangle$DEF$=n.p$; and $s$ the angle-sum of any triangle equivalent to $p$; manifestly

$$\text{st.}\angle-u=m.s-(m-1)\text{st.}\angle=\text{st.}\angle-m(\text{st.}\angle-s);$$

and $u=m(\text{st.}\angle-s)$; and in like manner $v=n(\text{st.}\angle-s)$.

Therefore $\triangle$ABC : $\triangle$DEF$=m:n=u:v$.

It is evidently also easily extended to the case of the incommensurability of the triangles ABC, DEF.

In the same way is demonstrated that triangles on a sphere are as the *excesses* of the sums of their angles above a st.$<$.

If two angles of the spherical $\triangle$ are right, the third $z$ will be the said *excess*.   But

(a great circle being called $p$) this $\triangle$ is manifestly

$$=\frac{z}{2\pi}\frac{p^2}{2\pi}\ (\S\ 32,\ \mathrm{VI});$$

consequently, any triangle of whose angles the excess is $z$, is

$$=\frac{zp^2}{4\pi^2}.$$

§ 43. Now, in S, the area of a rectilineal $\triangle$ is expressed by means of the sum of its angles.



FIG. 39.

If AB increases to infinity; (§ 42)  $\triangle\,\mathrm{ABC} : (\mathrm{rt}.\angle-u-v)$ will be constant. But $\triangle\,\mathrm{ABC}$ $=\mathrm{BACN}$ (§ 32, V), and $\mathrm{rt}.\angle$ $-u-v=z$  (§ 1);  and so $\mathrm{BACN} : z = \triangle\,\mathrm{ABC} : (\mathrm{rt}.\,\angle-u-v)=\mathrm{BAC'N'} : z'.$

Moreover, manifestly (§ 30) $\mathrm{BDCN} : \mathrm{BD'C'N'}=r : r' = \tan z : \tan z'.$

But for $y'=$o, we have

$$\frac{\mathrm{BD'C'N'}}{\mathrm{BAC'N'}}=1,\ \text{and also}\ \frac{\tan z'}{z'}=1;$$

consequently,

$$\mathrm{BDCN} : \mathrm{BACN}=\tan z : z.$$

But (§ 32)

$$\mathrm{BDCN}=r.i=i^2\tan z;$$

therefore,      $\mathrm{BACN}=z.i^2.$

Designating henceforth, for brevity, any tri-
angle the supplement of whose angle-sum is $z$
by $\triangle$, we will therefore have $\triangle = z.i^2$.


FIG. 40.

Hence it readily flows
that, if OR∥AM and
RO∥AB, the *area* com-
prehended between the
straights OR, ST, BC [25]
(which is manifestly the
absolute limit of the area of rectilineal tri-
angles increasing without bound, or of $\triangle$ for
$z=$st. $\angle$), is $=\pi i^2=$ area $\odot i$, in F.

This limit being denoted by $\square$, moreover
(by § 30) $\pi r^2 = \tan^2 z . \square =$ area $\odot r$ in F (§ 21)$=$
area $\odot s$ (by §32, VI) if the chord CD is called $s$.

If now, bisecting at right angles the given
radius $s$ of the circle in a plane (or the L form
radius of the circle in F), we construct (by
§ 34) DB∥$\rightleftharpoons$CN; by dropping CA $\perp$ DB, and


FIG. 41.

erecting CM $\perp$ CA, we shall
get $z;$ whence (by § 37), assum-
ing at pleasure an L form
radius for unity, *tan²z can be
determined geometrically by
means of two uniform lines
of the same curvature* (which,
their extremities alone being
given and their axes con-

structed, manifestly may be compared like straights, and in this respect considered equivalent to straights).

Moreover, a quadrilateral, ex. gr. regular $= \square$ is constructed as follows:


FIG. 42.

Take ABC=rt.$\angle$, BAC=$\frac{1}{2}$ rt. $\angle$, ACB=$\frac{1}{4}$ rt. $\angle$, and BC=$x$.

By mere square roots, X (from § 31, II) can be expressed and (by § 37) constructed; and having X (by § 38 or also §§ 29 and 35), $x$ itself can be determined. And octuple $\triangle$ ABC is manifestly $= \square$, and by this *a plane circle of radius s is geometrically squared by means of a rectilinear figure and uniform lines of the same species* (equivalent to straights as to comparison *inter se*); *but an F form circle is planified in the same manner: and we have either the Axiom* XI *of Euclid true or the geometric quadrature of the circle*, although thus far it has remained undecided, which of these two has place in reality.

Whenever $\tan^2 z$ is either a whole number, or a rational fraction, whose denominator (reduced to the simplest form) is either a prime number of the form $2^m+1$ (of which is also $2=2^0+1$), or a product of however many prime numbers of this form, of which each (with the

exception of 2, which alone may occur any number of times) occurs *only once* as factor, we can, by the theory of polygons of the illustrious Gauss (remarkable invention of our, nay of every age) (and only for such values [26] of $z$), construct a rectilineal figure $=\tan^2 z \square =$ area $\odot s$.  For the division of $\square$ (the theorem of § 42 extending easily to any polygons) manifestly requires the partition of a st. $\angle$, which (as can be shown) can be achieved geometrically only under the said condition.

But in all such cases, what precedes conducts easily to the desired end.   And any rectilineal figure can be converted geometrically into a regular polygon of $n$ sides, if $n$ falls under the Gaussian form.

It remains, finally (that the thing may be completed in every respect), to demonstrate the impossibility (apart from any supposition), of deciding *a priori*, whether $\Sigma$, or some S (and which one) exists.   This, however, is reserved for a more suitable occasion.

# APPENDIX I.

---

Finally it may be permitted to add something appertaining to the author of the *Appendix* in the first volume, who, however, may pardon me if something I have not touched with his acuteness.

The thing consists briefly in this: *the formulas of spherical trigonometry* (demonstrated in the said *Appendix* independently of Euclid's Axiom XI) *coincide with the formulas of plane trigonometry*, *if* (in a way provisionally speaking) *the sides of a spherical triangle are accepted as reals, but of a rectilineal triangle as imaginaries;* so that, as to trigonometric formulas, the plane may be considered as an imaginary sphere, if for real, that is accepted in which sin rt. $\angle = 1$.

Doubtless, of the Euclidean axiom has been said in volume first enough and to spare: for

the case if it were not true, is demonstrated
(Tom. I. App., p. 13), that there is given a cer-
tain $i$, for which the I there mentioned is $=e$
(the base of natural logarithms), and for this
case are established also (*ibidem,* p. 14) the
formulas of plane trigonometry, and indeed so,
that (by the side of p. 19, ibidem) the formulas
are still valid for the case of the verity of the
said axiom; indeed if the limits of the values
are taken, supposing that $i \doteq \infty$; truly the
Euclidean system is as if the limit of the anti-
Euclidean (for $i \doteq \infty$).

Assume for the case of $i$ existing, the unit
$= i$, and extend the concepts sine and cosine
also to imaginary arcs, so that, $p$ designating
an arc whether real or imaginary,

$$\frac{e^{p\sqrt{-1}} + e^{-p\sqrt{-1}}}{2} \text{ is called the}$$

*cosine* of $p$, and

$$\frac{e^{p\sqrt{-1}} - e^{-p\sqrt{-1}}}{2\sqrt{-1}} \text{ is called}$$

the *sine* of $p$ (as Tom. I., p. 177).

Hence for $q$ real

$$\frac{e^{q} - e^{-q}}{2\sqrt{-1}} = \frac{e^{-q\sqrt{-1}\cdot\sqrt{-1}} - e^{q\sqrt{-1}\cdot\sqrt{-1}}}{2\sqrt{-1}} = \sin(-q\sqrt{-1})$$
$$= -\sin(q\sqrt{-1}).$$

So $\quad \dfrac{e^{q}+e^{-q}}{2} = \dfrac{e^{-q\sqrt{-1}\cdot\sqrt{-1}}+e^{q\sqrt{-1}\cdot\sqrt{-1}}}{2} = \cos(-q\sqrt{-1})$

$$= \cos(q\sqrt{-1});$$

if of course also in the imaginary circle, the sine of a negative arc is the same as the sine of a positive arc otherwise equal to the first, except that it is negative, and the cosine of a positive arc and of a negative (if otherwise they be equal) the same.

In the said *Appendix*, § 25, is demonstrated absolutely, that is, independently of the said axiom; that, in any rectilineal triangle *the sines of the circles are as the circles of radii equal to the sides opposite.*

Moreover is demonstrated for the case of $i$ existing, that the circle of radius $y$ is

$$= \pi i \left[ e^{\frac{y}{i}} - e^{\frac{-y}{i}} \right], \text{ which, for } i=1, \text{ becomes}$$

$$\pi(e^{y}-e^{-y}).$$

Therefore (§ 31 *ibidem*), for a right-angled rectilineal triangle of which the sides are $a$ and $b$, the hypothenuse $c$, and the angles opposite to the sides $a, b, c$ are $\alpha$, $\beta$, rt. $\angle$, (for $i=1$), in I,

$$1 : \sin \alpha = \pi(e^{c}-e^{-c}) : \pi(e^{a}-e^{-a});$$

and so

$$1 : \sin\alpha = \frac{e^{c}-e^{-c}}{2\sqrt{-1}} : \frac{e^{a}-e^{-a}}{2\sqrt{-1}}. \qquad \text{Whence } 1 : \sin \alpha$$

$$= -\sin\ (c\sqrt{-1}) : -\sin\ (a\sqrt{-1}).$$    And hence

$$1 : \sin\ a = \sin\ (c\sqrt{-1}) : \sin\ (a\sqrt{-1}).$$

In II becomes

$$\cos\ a : \sin\ \beta = \cos\ (a\sqrt{-1}) : 1;$$

in III becomes

$$\cos\ (c\sqrt{-1}) = \cos\ (a\sqrt{-1}) . \cos\ (b\sqrt{-1}).$$

These, as all the formulas of plane trigonometry deducible from them, coincide completely with the formulas of spherical trigonometry; except that if, ex. gr., also the sides and the angles opposite them of a right-angled spherical triangle and the hypothenuse bear the same names, the sides of the rectilineal triangle are to be divided by $\sqrt{-1}$ to obtain the formulas for the spherical triangle.

Obviously we get (clearly as Tom., II., p. 252),

from I,        $1 : \sin\ a = \sin\ c : \sin\ a;$

from II,        $1 : \cos\ a = \sin\beta : \cos\ a;$

from III,        $\cos\ c = \cos\ a \cos\ b.$

Though it be allowable to pass over other things; yet I have learned that the reader may be offended and impeded by the deduction omitted, (Tom. I., App., p. 19) [in § 32 at end]: it will not be irrelevant to show how, ex. gr., from

$$e^{\frac{c}{i}} + e^{\frac{-c}{i}} = \tfrac{1}{2} \left( e^{\frac{a}{i}} + e^{\frac{-a}{i}} \right) \left( e^{\frac{b}{i}} + e^{\frac{-b}{i}} \right)$$

follows

$$c^2 = a^2 + b^2.$$

(the theorem of Pythagoras for the Euclidean system); probably thus also the author deduced it, and the others also· follow in the same manner.

Obviously we have, the powers of $e$ being expressed by series (like Tom. I., p. 168),

$$e^{\frac{k}{i}} = 1 + \frac{k}{i} + \frac{k^2}{2i^2} + \frac{k^3}{2.3.i^3} + \frac{k^4}{2.3.4.i^4} \cdot \cdot \cdot \cdot \cdot ,$$

$$e^{-\frac{k}{i}} = 1 - \frac{k}{i} + \frac{k^2}{2i^2} - \frac{k^3}{2.3.i^3} + \frac{k^4}{2.3.4.i^4} \cdot \cdot \cdot \cdot , \text{ and so}$$

$$e^{\frac{k}{i}} + e^{\frac{-k}{i}} = 2 + \frac{k^2}{i^2} + \frac{k^4}{3.4.i^4} + \frac{k^6}{3.4.5.6.i^6} \cdot \cdot \cdot \cdot \cdot ,$$

$$= 2 + \frac{k^2 + u}{i^2}, \text{ (designating by}$$

$\frac{u}{i^2}$ the sum of all the terms after $\frac{k^2}{i^2}$) ; and we have $u \doteq 0$, while $i \doteq \infty$. For all the terms which follow $\frac{k^2}{i^2}$ are divided by $i^2$; the first term will be $\frac{k^4}{3.4i^2}$; and any ratio $< \frac{k^2}{i^2}$; and though the ratio everywhere should remain this, the sum would be (Tom. I., p. 131),

$$\frac{k^4}{3.4.i^2} : \left[ 1 - \frac{k^2}{i^2} \right] = \frac{k^4}{3.4.(i^2 - k^2)},$$

which manifestly $\doteq 0$, while $i \doteq \infty$.

And from

$$e^{\frac{c}{i}}+e^{\frac{-c}{i}}=\tfrac{1}{2}\left(e^{\frac{(a+b)}{i}}+e^{\frac{-(a+b)}{i}}+e^{\frac{a-b}{i}}+e^{\frac{-(a-b)}{i}}\right)$$

follows (for $w$, $v$, $\lambda$ taken like $u$)

$$2+\frac{c^2+w}{i^2}=\tfrac{1}{2}\left[\cdot 2+\frac{(a+b)^2+v}{i^2}+2+\frac{(a+b)^2+\lambda}{i^2}\right]\cdot$$

And hence

$$c^2=\frac{a^2+2ab+b^2+a^2-2ab+b^2+v+\lambda-w}{2},$$

which $\dot{=}a^2+b^2$.

# APPENDIX II.

---

Lobachevski and the author of the *Appendix* each consider two points A, B, of the sphere-limit, and the corresponding axes ray AM, ray BN (§ 23).



FIG. 43.

They demonstrate that, if $\alpha$, $\beta$, $\gamma$ designate the arcs of the circle limit AB, CD, HL, separated by segments of the axis AC=1, AH =$x$, we have

$$\frac{\alpha}{\gamma} = \left( \frac{\alpha}{\beta} \right)^{x}.$$

Lobachevski represents the value of $\frac{\gamma}{\alpha}$ by $e^{-x}$, $e$ having some value $>1$, dependent on the unit for length that we have chosen, and able to be supposed equal to the Naperian base.

The author of the *Appendix* is led directly to introduce the base of natural logarithms.

If we put $\frac{a}{\beta}=\delta$, and $\gamma$, $\gamma'$ are arcs situated at the distances $y$, $i$ from $a$, we shall have

$$\frac{a}{\gamma}=\delta^y=\mathrm{Y}, \qquad \frac{a}{\gamma'}=\delta^i=\mathrm{I}, \quad \text{whence} \quad \mathrm{Y}=\mathrm{I}^{\frac{y}{i}}.$$

He demonstrates afterward (§ 29) that, if $u$ is the angle which a straight makes with the perpendicular $y$ to its parallel, we have

$$\mathrm{Y}=\cot \tfrac{1}{2}u.$$

Therefore, if we put $z=\frac{\pi}{2}-u$, we have

$$\mathrm{Y}=\tan\,(z+\tfrac{1}{2}u)=\frac{\tan z+\tan\tfrac{1}{2}u}{1-\tan z\,\tan\tfrac{1}{2}u},$$

whence we get, having regard to the value of $\tan \tfrac{1}{2}u=\mathrm{Y}^{-1}$,

$$\tan z=\tfrac{1}{2}\,(\mathrm{Y}-\mathrm{Y}^{-1})=\tfrac{1}{2}\left[\mathrm{I}^{\frac{y}{i}}-\mathrm{I}^{\frac{-y}{i}}\right]\;(\S 30).$$

If now $y$ is the semi-chord of the arc of circle-limit $2r$, we prove (§ 30) that $\dfrac{r}{\tan z}=$ constant.

Representing this constant by $i$, and making $y$ tend toward zero, we have

$$\frac{2r}{2y}\doteq 1, \quad \text{whence}$$

$$2y\doteq 2\,i\,\tan z\doteq i\,\frac{\mathrm{I}^{\frac{2y}{i}}-1}{\mathrm{I}^{\frac{y}{i}}},$$

or putting $\frac{2y}{i}=k$, I$=el$,

$$kI^{\frac{y}{i}\cdot}=e^{kl}-1=kl\ (1+\lambda),$$

$\lambda$ being infinitesimal at the same time as $k$. Therefore, for the limit, $1=l$ and consequently I$=e$.

The circle traced on the sphere-limit with the arc $r$ of the curve-limit for radius, has for length $2\pi r$.   Therefore,

$$\odot y=2\pi r=2\pi i\ \tan z=\pi i\ (Y-Y^{-1}).$$

In the rectilineal $\triangle$ where $\alpha$, $\beta$ designate the angles opposite the sides $a$, $b$, we have (§ 25)

$$\sin\ \alpha:\sin\ \beta=\odot a:\odot b=\pi i(A-A^{-1}):\pi i(B-B^{-1})$$
$$=\sin\ (a^{\sqrt{-1}}):\sin\ (b^{\sqrt{-1}}).$$

Thus in plane trigonometry as in spherical trigonometry, the sines of the angles are to each other as the sines of the opposite sides, only that on the sphere the sides are reals, and in the plane we must consider them as imaginaries, just as if the plane were an imaginary sphere.

We may arrive at this proposition without a preceding determination of the value of I.

If we designate the constant $\dfrac{r}{\tan z}$ by $q$, we shall have, as before

$$\odot y=\pi q\ (Y-Y^{-1}),$$

whence we deduce the same proportion as above, taking for $i$ the distance for which the ratio I is equal to $e$.

If axiom *XI* is not true, there exists a determinate $i,$ which must be substituted in the formulas.

If, on the contrary, this axiom is true, we must make in the formulas $i = \infty$. Because, in this case, the quantity $\frac{a}{\gamma} = Y$ is always $= 1$, the sphere-limit being a plane, and the axes being parallel in Euclid's sense.

The exponent $\frac{y}{i}$ must therefore be zero, and consequently $i = \infty$.

It is easy to see that Bolyai's formulas of plane trigonometry are in accord with those of Lobachevski.

Take for example the formula of § 37,

$$\tan \text{\textit{//}} (a) = \sin \text{B} \tan \text{\textit{//}} (p),$$

$a$ being the hypothenuse of a right-angled triangle, $p$ one side of the right angle, and B the angle opposite to this side.

Bolyai's formula of § 31, I, gives

$$1 : \sin \text{B} = (\text{A} - \text{A}^{-1}) : (\text{P} - \text{P}^{-1}).$$

Now, putting for brevity, $\frac{1}{2} \text{\textit{//}} (k) = k'$, we have $\tan 2p' : \tan 2a' = (\cot a' - \tan a') : (\cot p' - \tan p') = (\text{A} - \text{A}^{-1}) : (\text{P} - \text{P}^{-1}) = 1 : \sin \text{B}.$

# APPENDIX III.

---

## LIGHT FROM NON-EUCLIDEAN SPACES ON THE TEACHING OF ELEMENTARY GEOMETRY.

### By G. B. Halsted.

As foreshadowed by Bolyai and Riemann, founded by Cayley, extended and interpreted for hyperbolic, parabolic, elliptic spaces by Klein, recast and applied to mechanics by Sir Robert Ball, *projective metrics* may be looked upon as characteristic of what is highest and most peculiarly modern in all the bewildering range of mathematical achievement.

Mathematicians hold that number is wholly a creation of the human intellect, while on the contrary our space has an empirical element. Of possible geometries we can not say *a priori* which shall be that of our actual space, the space in which we move. Of course an advance so important, not only for mathematics but for philosophy, has had some metaphysical opponents, and as long ago as 1878 I mentioned in my Bibliography of Hyper-

Space and Non-Euclidean Geometry (American Journal of Mathematics, Vol. I, 1878, Vol. II, 1879) one of these, Schmitz-Dumont, as a sad paradoxer, and another, J. C. Becker, both of whom would ere this have shared the oblivion of still more antiquated fighters against the light, but that Dr. Schotten, praiseworthy for the very attempt at a comparative planimetry, happens to be himself a believer in the *a priori* founding of geòmetry, while his American reviewer, Mr. Ziwet, was then also an anti-non-Euclidean, though since converted.

He says, ''we find that some of the best German text books do not try at all to define what is space, or what is a point, or even what is a straight line.'' Do any German geometries define space? I never remember to have met one that does.

In experience, what comes first is a bounded surface, with its boundaries, lines, and their boundaries, points. Are the points whose definitions are omitted anything different or better?

Dr. Schotten regards the two ideas ''direction'' and ''distance'' as intuitively given in the mind and as so simple as to not require definition.

When we read of two jockeys speeding

around a track in opposite directions, and
also on page 87 of Richardson's Euclid, 1891,
read, "The sides of the figure must be pro-
duced in the same direction of rotation; . . .
going round the figure always in the same
direction," we do not wonder that when Mr.
Ziwet had written: "he therefore bases the
definition of the straight line on these two
ideas," he stops, modifies, and rubs that out
as follows, "or rather recommends to eluci-
date the intuitive idea of the straight line
possessed by any well-balanced mind by means
of the still simpler ideas of direction" [in a
circle] "and distance" [on a curve].

But when we come to geometry as a science,
as foundation for work like that of Cayley and
Ball, I think with Professor Chrystal: "It is
essential to be careful with our definition of a
*straight line*, for it will be found that vir-
tually the properties of the straight line de-
termine the nature of space.

"Our definition shall be that two points *in
general* determine a straight line."

We presume that Mr. Ziwet glories in that
unfortunate expression "a straight line is the
shortest distance between two points," still
occurring in Wentworth (New Plane Geom-
etry, page 33), even after he has said, page 5,

"the length of the straight line is called the *distance* between two points." If the *length* of the one straight line between two points is the distance between those points, how can the straight line itself be the *shortest* distance? If there is only one distance, it is the longest as much as the shortest distance, and if it is the *length* of this shorto-longest distance which is the distance then it is not the straight line itself which is the longo-shortest distance. But Wentworth also says: "Of all lines joining two points the *shortest* is the straight line."

This general comparison involves the measurement of curves, which involves the theory of limits, to say nothing of ratio. The very ascription of length to a curve involves the idea of a limit. And then to introduce this general axiom, as does Wentworth, only to prove a very special case of itself, that two sides of a triangle are together greater than the third, is surely bad logic, bad pedagogy, bad mathematics.

This latter theorem, according to the first of Pascal's rules for demonstrations, should not be proved at all, since every dog knows it. But to this objection, as old as the sophists, Simson long ago answered for the science of

geometry, that the number of assumptions ought not to be increased without necessity; or as Dedekind has it: "*Was beweisbar ist, soll in der Wissenschaft nicht ohne Beweis geglaubt werden.*"

Professor W. B. Smith (Ph. D., Goettingen), has written: "Nothing could be more unfortunate than the attempt to lay the notion of Direction at the bottom of Geometry."

Was it not this notion which led so good a mathematician as John Casey to give as a demonstration of a triangle's angle-sum the procedure called "a *practical* demonstration" on page 87 of Richardson's Euclid, and there described as "laying a 'straight edge' along one of the sides of the figure, and then turning it round so as to coincide with each side in turn."

This assumes that a segment of a straight line, a sect, may be translated without rotation, which assumption readily comes to view when you try the procedure in two-dimensional spherics. Though this fallacy was exposed by so eminent a geometer as Olaus Henrici in so public a place as the pages of 'Nature,' yet it has just been solemnly reproduced by Professor G. C. Edwards, of the University of California, in his Elements of Geometry: Mac-

Millan, 1895. It is of the greatest importance for every teacher to know and connect the commonest forms of assumption equivalent to Euclid's Axiom XI. If in a plane two straight lines perpendicular to a third nowhere meet, are there others, not both perpendicular to any third, which nowhere meet? Euclid's Axiom XI is the assumption *No*. Playfair's answers *no* more simply. But the very same answer is given by the common assumption of our geometries, usually unnoticed, that a circle may be passed through any three points not costraight.

This equivalence was pointed out by Bolyai Farkas, who looks upon this as the simplest form of the assumption. Other equivalents are, the existence of any finite triangle whose angle-sum is a straight angle; or the existence of a plane rectangle; or that, in triangles, the angle-sum is constant.

One of Legendre's forms was that through every point within an angle a straight line may be drawn which cuts both arms.

But Legendre never saw through this matter because he had not, as we have, the eyes of Bolyai and Lobachevski to see with. The same lack of their eyes has caused the author of the charming book "Euclid and His Modern

Rivals,'' to give us one more equivalent form:
''In any circle, the inscribed equilateral tetra-
gon is greater than any one of the segments
which lie outside it.'' (A New Theory of
Parallels by C. L. Dodgson, 3d. Ed., 1890.)

Any attempt to define a straight line by
means of ''direction'' is simply a case of ''ar-
gumentum in circulo.'' In all such attempts
the loose word ''direction'' is used in a sense
which presupposes the straight line. The
directions from a point in Euclidean space are
only the $\infty^2$ rays from that point.

Rays not costraight can be said to have the
same direction only after a theory of parallels
is presupposed, assumed.

Three of the exposures of Professor G. C.
Edwards' fallacy are here reproduced. The
first, already referred to, is from *Nature,* Vol.
XXIX, p. 453, March 13, 1884.

''I select for discussion the 'quaternion
proof'' given by Sir William Hamilton. . . .
Hamilton's proof consists in the following:

''One side AB of the triangle ABC is turned
about the point B till it lies in the continuation
of BC; next, the line BC is made to slide along
BC till B comes to C, and is then turned about
C till it comes to lie in the continuation of AC.

"It is now again made to slide along CA till the point B comes to A, and is turned about A till it lies in the line AB.  Hence it follows, *since rotation is independent of translation,* that the line has performed a whole revolution, that is, it has been turned through four right angles.  But it has also described in succession the three exterior angles of the triangle, hence these are together equal to four right angles, and from this follows at once that the interior angles are equal to two right angles.

"To show how erroneous this reasoning is— in spite of Sir William Hamilton and in spite of quaternions—I need only point out that it holds exactly in the same manner for a triangle on the surface of the sphere, from which it would follow that the sum of the angles in a spherical triangle equals two right angles, whilst this sum is known to be always greater than two right angles.  The proof depends only on the fact, that any line can be made to coincide with any other line, that two lines do so coincide when they have two points in common, and further, that a line may be turned about any point in it without leaving the surface.  But if instead of the plane we take a spherical surface, and instead of a line a great

circle on the sphere, all these conditions are again satisfied.

"The reasoning employed must therefore be fallacious, and the error lies in the words printed in italics; for these words contain an assumption which has not been proved.

"O. HENRICI."

Perronet Thompson, of Queen's College, Cambridge, in a book of which the third edition is dated 1830, says:
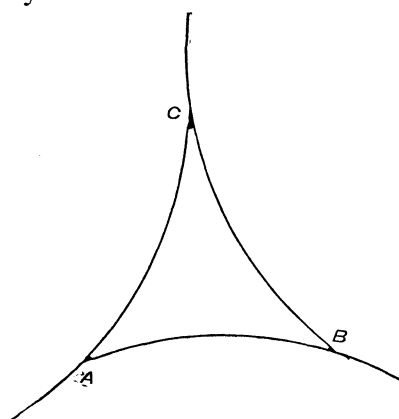
"Professor Playfair, in the Notes to his 'Elements of Geometry' [1813], has proposed another demonstration, founded on a remarkable *non causa pro causa*.

"It purports to collect the fact [Eu. I., 32, Cor., 2] that (on the sides being successively prolonged to the same hand) the exterior angles of a rectilinear triangle are together equal to four right angles, from the circumstance that a straight line carried round the perimeter of a triangle by being applied to all the sides in succession, is brought into its old situation again; the argument being, that because this line has made the sort of somerset it would do by being turned through four right angles about a fixed point, the exterior

angles of the triangle have necessarily been equal to four right angles.

"The answer to which is, that there is no connexion between the things at all, and that the result will just as much take place where the exterior angles are avowedly not equal to four right angles.

"Take, for example, the plane triangle formed by three small arcs of the same or equal circles, as in the margin; and it is manifest that an arc of this circle may be carried round precisely in the way described and return to its old situation, and yet there be no pretense for inferring that the exterior angles were equal to four right angles.

"And if it is urged that these are *curved* lines and the statement made was of straight; then the answer is by demanding to know, what property of straight lines has been laid down or established, which determines that what is not true in the case of other lines is

true in theirs. It has been shown that, as a general proposition, the connexion between a line returning to its place and the exterior angles having been equal to four right angles, is a *non sequitur;* that it is a thing that may be or may not be; that the notion that it returns to its place *because* the exterior angles have been equal to four right angles, is a mistake. From which it is a legitimate conclusion, that if it had pleased nature to make the exterior angles of a triangle greater or less than four right angles, this would not have created the smallest impediment to the line's returning to its old situation after being carried round the sides; and consequently the line's returning is no evidence of the angles not being greater or less than four right angles.''

Charles L. Dodgson, of Christ Church, Oxford, in his ''Curiosa Mathematica,'' Part I, pp. 70–71, 3d Ed., 1890, says:

''Yet another process has been invented—quite fascinating in its brevity and its elegance—which, though involving the same fallacy as the Direction-Theory, proves Euc. I, 32, without even mentioning the dangerous word 'Direction.'
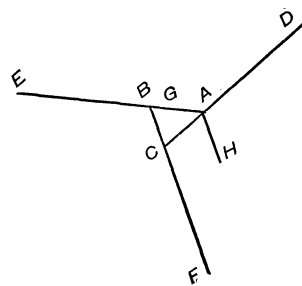
"We are told to take any triangle A B C; to produce C A to D; to make part of CD, viz., AD, revolve, about A, into the position A B E; then to make part of this line, viz., BE, revolve, about B, into the position BCF; and lastly to make part of this line, viz., CF, revolve, about C, till it lies along CD, of which it originally formed a part. We are then assured that it must have revolved through four right angles: from which it easily follows that the interior angles of the triangle are together equal to two right angles.

"The disproof of this fallacy is almost as brief and elegant as the fallacy itself. We first quote the general principle that we can not reasonably be told to make a line fulfill *two* conditions, either of which is enough by itself to fix its position: e. g., given three points X, Y, Z, we can not reasonably be told to draw a line from X which shall pass through Y *and* Z: we can make it pass through Y, but it must then take its chance of passing through Z; and *vice versa*.

"Now let us suppose that, while one part of

AE, viz., BE, revolves into the position BF, another little bit of it, viz., AG, revolves, through an equal angle, into the position AH; and that, while CF revolves into the position of lying along CD, AH revolves—and here comes the fallacy.

"You must not say 'revolves, through an equal angle, into the position of lying along AD,' for this would be to make AH *fulfill two conditions at once*.

"If you say that the one condition involves the other, you are virtually asserting that the lines CF, AH are equally inclined to CD—and this in *consequence* of AH having been so drawn that these same lines are equally inclined to AE.

"That is, you are asserting, 'A pair of lines which are equally inclined to a certain transversal, are so to any transversal.'  [Deducible from Euc. I, 27, 28, 29.]"

# MATHEMATICAL WORKS

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